TOURNAMENT OF THE TOWNS, 2003–2004

Training Session, 8 November 2003

JUNIOR QUESTIONS: Years 8, 9, 10

1. Find the locus of points M inside the rhombus $ABCD$ such that the sum of the angles $\angle AMB$ and $\angle CMD$ equals $180°$. (5 points)

   (**Definition** A *locus* is a set of points that

   (i) all points in the set satisfy a given condition, and

   (ii) all points that satsify the given condition belong to the set.)

2. Two circles c and d are situated in the plane each outside the other. The points C and D are located on the circles c and d respectively, so as to be as far apart as possible. Two smaller circles are constructed inside c and d. Of these the first circle touches c and the two tangents drawn from C to d, while the second circle touches d and the two tangents drawn from D to c. Prove that the smaller circles are equal. (J. Tabov, Sofia, 4 points)

3. There are 68 coins, each coin having a different weight than that of each other. Show how to find the heaviest and lightest coins in 100 weighings on a beam balance. (S. Fomin, Leningrad, 5 points)

4. In a triangle ABC, $|AB| = |BC|$. K is a point on AB and L is a point on BC such that $|AK| + |LC| = |KL|$. A line through the midpoint M of KL and parallel to BC intersects AC at the point N. Determine $\angle KNL$. (Northern Spring, 2003, Junior paper, 5 points)

5. We are given two three-litre bottles, one containing 1 litre of water and the other containing 1 litre of 2% salt solution. One can pour liquids from one bottle to the other and then mix them to obtain solutions of different concentrations. Can one obtain a 1.5% solution of salt in the bottle which originally contained water? (S. Fomin, Leningrad, 3 points)

6. There are 36 cards in a deck arranged in the sequence spades, clubs, hearts, diamonds, spades, clubs, hearts, diamonds, etc. Somebody took part of this deck off the top, turned it upside down, and cut this part into the remaining part of the deck (i.e. inserted it between two consecutive cards). Then four cards were taken of the top, then another four etc. Prove that in any of these sets of four cards, all the cards are of different suits. (A. Merkov, Moscow, 12 points)

SENIOR QUESTIONS: Years 11, 12

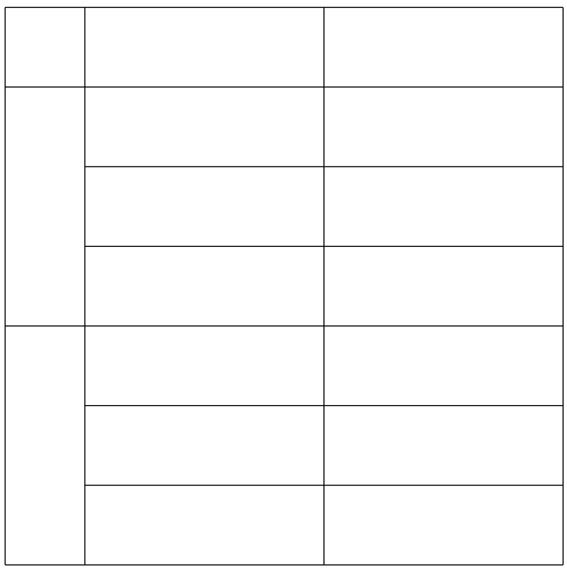
1. Prove than an $a \times b$ rectangle, where $\frac{b}{2} < a < b$, can be cut into three pieces from which a square may be put together. (Northern Spring, 2003, Junior paper, 5 points)

2. A 7×7 square is made up of sixteen 1×3 tiles and one 1×1 tile. Prove that the 1×1 tile lies either at the centre of the square or adjoins one of its boundaries. (10 points)

   Solution. Partition the 7×7 square into a 7×7 array of 1×1 squares and label the smaller squares cyclically as follows

$$\begin{array}{ccccc} A & B & C & A & \cdots \\ B & C & A & B & \\ C & A & B & C & \\ A & B & C & A & \\ \vdots & & & & \ddots \end{array}$$

   Each 1×3 tile necessarily covers adjoining squares, i.e. squares labelled A, B and C. There are 17 A squares, 16 Bs and 16 Cs. Thus the 1×1 tile must occupy a square labelled A. However, since the orientation of the square is not relevant, for feasible locations of the 1×1 tile we should eliminate all tiles labelled A which do not remain as As when the 7×7 square is rotated through multiples of 90° or reflected in an axis of symmetry. This leaves only those As in the corners, those at the midpoints of each edge and the centre square. The diagram below shows one configuration where the 1×1 tile is in a corner. Rotations of the top 4×4 square through 90° or 180° give configurations where the 1×1 tile occupies the middle of an edge or the middle of the 7×7 square, respectively. Hence all the placements of the 1×1 tile that were shown to be feasible are indeed possible, and hence we have shown what we were required to prove.



3. Each of the numbers $1, 2, 3, \ldots, 25$ is arranged in a 5 by 5 table. In each row they appear in increasing order (left to right). Find the maximal and minimal possible sum of the numbers in the third column. (Folklore, 5 points)

Solution. Denote by a_{ij} the number in the ith row and jth column. We are given that $a_{ij} < a_{ik}$, if $j < k$. W.l.o.g. we may assume that $a_{i3} < a_{\ell 3}$, if $i < \ell$. Since $a_{53} < a_{54} < a_{55}$, a_{53} is at most 23. Since $a_{43} < a_{44} < a_{45}$ and $a_{43} < a_{53}$ there are at least five table entries, namely $a_{44}, a_{45}, a_{53}, a_{54}, a_{55}$, larger than a_{43}, and so a_{43} is at most 20. Similarly, at least eight table entries are larger than a_{33}, and so on. Thus we have that $a_{33} \leq 17, a_{23} \leq 14$ and $a_{13} \leq 11$. Hence the sum of the five entries in column 3 of the table is at most

$$23 + 20 + 17 + 14 + 11 = 85.$$

By an analogous argument $a_{11} < a_{12} < a_{13}$, etc. and so we have that $a_{13} \geq 3, a_{23} \geq 6, a_{33} \geq 9, a_{43} \geq 12$ and $a_{53} \geq 15$, so that the sum of the entries in the table's 3rd column is at least

$$3 + 6 + 9 + 12 + 15 = 45.$$

Both these sums are attainable as shown below.

1	2	11	12	13
3	4	14	15	16
5	6	17	18	19
7	8	20	21	22
9	10	23	24	25

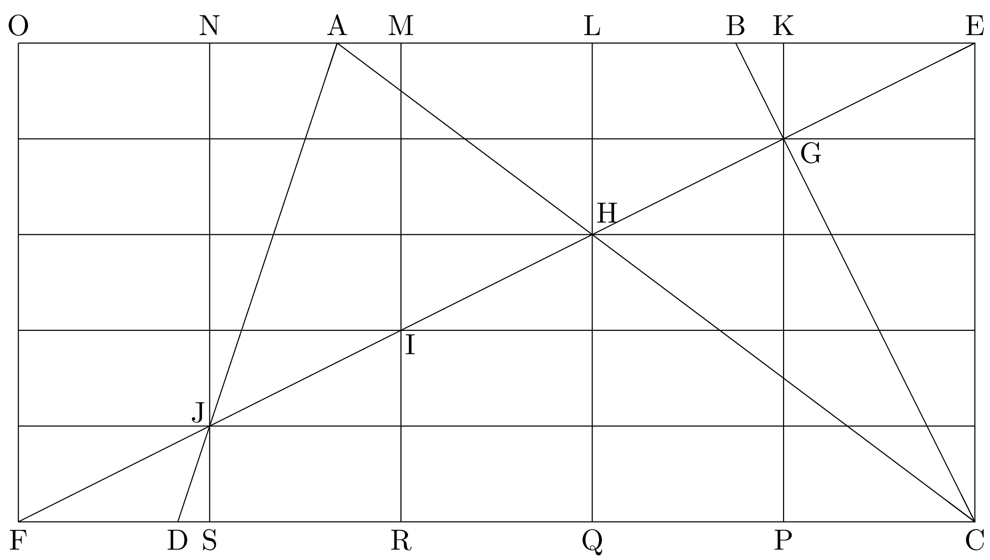
1	2	3	16	17
4	5	6	18	19
7	8	9	20	21
10	11	12	22	23
13	14	15	24	25

Thus the maximal and minimal possible sums of the numbers in the third column are 85 and 45, respectively. (Based on a proof by A. Liu)

4. In quadrilateral $ABCD$ it is given that $|AB| = |BC| = 1$, $\angle ABC = 100°$, and $\angle CDA = 130°$. Find the length of the line segment BD. (4 points)

5. Determine the ratio of the bases (parallel sides) of the trapezoid for which there exists a line with 6 points of intersection with the diagonals, lateral sides and extended bases cut 5 equal segments. (A. G. Gotman, 5 points)

Solution.



We construct a trapezoid $ABCD$ with the required properties as follows. Construct a parallelogram $OECF$ such that $|OE| = 5a$ units and $|OF| = 5b$ units (for simplicity we have drawn a rectangle, but the argument we give below works equally well if $OECF$ is a non-rectangular parallelogram). Construct a grid, by drawing 4 lines between and parallel to OF and EC equally spaced apart, i.e. at intervals of a units, and by drawing 4 lines between and parallel to OE and FC at intervals of b units. Draw EF; this will be the line with 6 points of intersection with the diagonals, lateral sides and extended bases of the trapezoid $ABCD$. By similar triangles, the grid cuts EF into 6 equal segments. Assign the points G, H, I, J, respectively to the internal intersections of EF with the grid as shown in the diagram. Draw CG to meet OE at B. Draw BI to meet FC at D, and draw DJ to meet OE at A. All that remains to be done is to determine the ratio $|CD|/|AB|$ and show that BD intersects EF at I. In order to do so we assign K, L, M, N to grid points between E and O, and P, Q, R, S to the grid points between F and C, as shown in the diagram.

Firstly we determine the ratio $|CD|/|AB|$. Since corresponding angles are equal (being either vertically opposite or alternate), triangles BKG and CPG are similar, and so $|BK|/|KG| = |CP|/|PG|$, i.e.

$$|BK| = \frac{|KG|}{|PG|}|CP| = \frac{b}{4b} \cdot a = \tfrac{1}{4}a$$

Since triangles ALH and CQH are similar, we have

$$\begin{aligned} \frac{|AL|}{|LH|} &= \frac{|CQ|}{|QH|} \\ |AL| &= \frac{|LH|}{|QH|}|CQ| \\ &= \frac{2b}{3b} \cdot 2a = \tfrac{4}{3}a \end{aligned}$$

Thus

$$|NA| = |NL| - |AL| = 2a - \tfrac{4}{3}a = \tfrac{2}{3}a$$

Now, since triangles SDJ and NAJ are similar, we have

$$\begin{aligned} \frac{|SD|}{|SJ|} &= \frac{|NA|}{|NJ|} \\ |SD| &= \frac{|SJ|}{|NJ|}|NA| \\ &= \frac{b}{4b} \cdot \tfrac{2}{3}a \\ &= \tfrac{1}{6}a \end{aligned}$$

Thus we have

$$\begin{aligned} |AB| &= |AL| + |LK| - |BK| \\ &= \tfrac{4}{3}a + a - \tfrac{1}{4}a \\ &= \tfrac{25}{12}a \\ |CD| &= |CS| + |SD| \\ &= 4a + \tfrac{1}{6}a \\ &= \tfrac{25}{6}a \end{aligned}$$

Thus $|CD|/|AB| = 2$.
Finally, suppose the line BD intersects MR at some point I'. We will show that in fact $I' = I$, by showing that I' cuts the line segment MR in the ratio $3 : 2$. Since triangles BMI' and FRI' are similar we have

$$\begin{aligned}\frac{|MI'|}{|RI'|} &= \frac{|MB|}{|RF|} \\ &= \frac{|MK| - |BK|}{|RF|} \\ &= \frac{2a - \frac{1}{4}a}{2a} \\ &= \tfrac{3}{2}\end{aligned}$$

Thus I' cuts MR in the ratio $3 : 2$, and hence $I' = I$.
Hence $ABCD$ is a trapezoid with the required properties and the ratio $|CD|/|AB|$ of the long base of the trapezoid to its short base is 2.

6. Prove that for all positive $a_1, a_2, \ldots, a_n$ the inequality

$$\left(1 + \frac{a_1^2}{a_2}\right)\left(1 + \frac{a_2^2}{a_3}\right)\cdots\left(1 + \frac{a_n^2}{a_1}\right) \geq (1 + a_1)(1 + a_2)\cdots(1 + a_n)$$

holds. (L. D. Kurliandchik, 5 points)

Solution. Define the proposition

$$P(n) : \left(1 + \frac{a_1^2}{a_2}\right)\left(1 + \frac{a_2^2}{a_3}\right)\cdots\left(1 + \frac{a_n^2}{a_1}\right) \geq (1 + a_1)(1 + a_2)\cdots(1 + a_n)$$

We prove $P(n)$ for all natural numbers n by induction. Firstly, $P(1)$ is true, since

$$\text{LHS of } P(1) = \left(1 + \frac{a_1^2}{a_1}\right) = (1 + a_1) = \text{RHS of } P(1)$$

Now we show $P(k) \implies P(k + 1)$, for any natural number k. In following through the inductive argument in the natural way we find that we would like

$$\left(1 + \frac{a_k^2}{a_{k+1}}\right)\left(1 + \frac{a_{k+1}^2}{a_1}\right) \geq \left(1 + \frac{a_k^2}{a_1}\right)(1 + a_{k+1})$$

which is equivalent to

$$(a_{k+1} - a_1)(a_{k+1}^2 - a_k^2) \geq 0$$

Observe that $P(k + 1)$ is invariant under the rotational permutation $1 \to 2 \to 3 \to \cdots \to k + 1 \to 1$. So we may assume that a_{k+1} is the largest of the a_i, and in particular that $a_{k+1} \geq a_1$ and $a_{k+1} \geq a_k$, from which we may deduce that

$$(a_{k+1} - a_1)(a_{k+1}^2 - a_k^2) \geq 0$$

and hence that

$$\left(1 + \frac{a_k^2}{a_{k+1}}\right)\left(1 + \frac{a_{k+1}^2}{a_1}\right) \geq \left(1 + \frac{a_k^2}{a_1}\right)(1 + a_{k+1}). \quad (1)$$

Now the argument is straightforward. By the inductive hypothesis we have $P(k)$, i.e.

$$\left(1+\frac{a_1^2}{a_2}\right)\left(1+\frac{a_2^2}{a_3}\right)\cdots\left(1+\frac{a_k^2}{a_1}\right) \geq (1+a_1)(1+a_2)\cdots(1+a_k). \tag{2}$$

Thus

$$\begin{aligned}
\text{LHS of } P(k+1) &= \left(1+\frac{a_1^2}{a_2}\right)\left(1+\frac{a_2^2}{a_3}\right)\cdots\left(1+\frac{a_k^2}{a_{k+1}}\right)\left(1+\frac{a_{k+1}^2}{a_1}\right) \\
&\geq \left(1+\frac{a_1^2}{a_2}\right)\left(1+\frac{a_2^2}{a_3}\right)\cdots\left(1+\frac{a_k^2}{a_1}\right)(1+a_{k+1}) && \text{by (1)} \\
&\geq (1+a_1)(1+a_2)\cdots(1+a_k)(1+a_{k+1}) && \text{by (2)} \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Thus we have shown $P(k+1)$ follows from $P(k)$.
So, by induction, $P(n)$ is true for all natural numbers n as was required to be shown.