TOURNAMENT OF THE TOWNS, 2003–2004

Training Session, 1 May, 2004

1. Show that the sum of the first n odd numbers is n^2?

2. For any positive integer n, consider the greatest odd divisor of the integers $n+1$ to $2n$ inclusive. Prove that sum of these n divisors is n^2.

3. An increasing arithmetic progression consists of one hundred positive integers. Is it possible that every two of them are relatively prime?

4. Show that if $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime decomposition of the positive integer n, then the number of divisors of n (including 1 and n) is $(e_1+1)(e_2+1)\cdots(e_k+1)$.

5. Which positive integers have exactly three positive divisors?

6. Which positive integers have exactly four positive divisors?

7. Show that a natural number n is an exact square if and only if it has an odd number of divisors.

*8. There are 50 prisoners in a row of locked cells. With the return of the King from the Crusades, a partial amnesty is declared and it works like this. When the prisoners are still asleep, the jailer walks past the cells 50 times, each time walking from left to right. On the first pass, he turns the lock in every cell (so that every cell is now open). On the second pass he turns the lock on every second cell (meaning that these cells are now locked again). On the third pass, he turns the lock on every third cell, and so on. In general, on the kth pass, he turns the lock on every kth cell. The question is: which cells are unlocked at the end of the process so that the prisoner is free to go?

9. Is the following statement true or false? *The number $n^2 + n + 41$ is prime for all positive integers n.*

10. Is the list of prime numbers *finite?* i.e. is there a *largest* prime number?

11. For each of the following pairs of integers a, b use the *Euclidean Algorithm* to find $d = (a, b)$ and find a pair of integers x, y such that $ax + by = d$.

    (i) $a = 85$, $b = 41$; (ii) $a = 2613$, $b = 637$.

12. Show that if there exist integers x, y such that $ax + by = 1$ then $(a, b) = 1$.

13. Show that $(3k+2, 5k+3) = 1$ for any integer k.

14. Show that $(a, a+2) = 2$ if a is even and $(a, a+2) = 1$ otherwise.

15. Show that if $(a, b) = 1$ then $(a+b, a-b) = 1$ or 2.

16. Find all solutions to the following *Diophantine Equations.*

    (i) $2x + 5y = 11$. (ii) $12x + 18y = 50$. (iii) $202x + 74y = 7638$.

    Does equation (iii) have a solution in *positive* integers x, y?

17. A grocer orders apples and oranges at a total cost of \$8.39. If apples cost 25c each and oranges cost 18c each, how many of each type of fruit did the grocer order?

18. An apartment block has units at two rates: most rent at \$87/week, but a few rent at \$123/week. When all are rented the gross income is \$8733/week. How many units of each type are there?

∗19. Find all integers x, y satisfying: $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{14}$.

∗20. When Jane is one year younger than Betty will be when Jane is half as old as Betty will be when Jane is twice as old as Betty is now, Betty will be three times as old as Jane was when Betty was as old as Jane is now.
One is in her teens and ages are in completed years. How old are they?

∗21. Prove that $5^{99} + 11^{99} + 17^{99}$ is divisible by 33.

∗22. What is the final digit of $(((((((((7^7)^7)^7)^7)^7)^7)^7)^7)^7)$? (7 occurs as a power 10 times.)