TOURNAMENT OF THE TOWNS, 2003–2004

Training Session, 24 April 2004

JUNIOR QUESTIONS: Years 8, 9, 10

1. A circle cuts each side of a rhombus twice thus dividing each side into three segments. Let us go around the perimeter of the rhombus clockwise beginning at a vertex and paint these segments successively in red, white and blue. Prove that the sum of the lengths of the blue segments equals that of the red ones.

(V. Proizvolov, Moscow, 4 points)

2. In a quadrilateral $ABCD$ the diagonals AC and BD meet at right angles, and angle $\angle ACD = \angle ACB$. Prove that $|AD| = |AB|$.

(3 points)

3. The line AB is parallel to the line CD, with A and D on the same side of BC, and $|BC| = |DC|$. Prove that BD bisects $\angle CBA$.

(3 points)

4. $ABCD$ is a parallelogram whose diagonals meet at E. If F, G, H and I are the midpoints of DE, AE, BE and CE, respectively, prove that $FGHI$ is a parallelogram.

(3 points)

5. In triangle ABC, the bisector of the angle at B meets AC at D and the bisector of the angle at C meets AB at E. These bisectors intersect at O and the lengths of OD and OE are equal. Prove that either $\angle BAC = 60°$ or triangle ABC is isosceles.

(6 points)

6. In a trapezoid $ABCD$ the sides BC and AD are parallel, $|AC| = |BC| + |AD|$, and the angle between the diagonals is equal to $\pi/3$. Prove that $|AB| = |CD|$.

(S. Smirnov, St. Petersburg, 3 points)

7. A, B, C and D are points outside a unit square $EFGH$ such that each of $\angle EAF$, $\angle FBG$, $\angle GCH$ and $\angle HDE$ is a right angle. O, P, Q and R are the incentres of triangles EAF, FBG, GCH and HDE respectively. Prove that

   (a) the area of the quadrilateral $ABCD$ does not exceed 2; and (3 points)

   (b) the area of the quadrilateral $OPQR$ does not exceed 1. (3 points)

(Northern Autumn, 2003, Senior A paper, 5 points)