TOURNAMENT OF THE TOWNS, 2003–2004

Chess Board and Similar Problems

Training Session, 10 April 2004

1. Coins are placed on two opposite corner squares of a chessboard. Is it possible to cover the rest of the chessboard with 2×1 dominoes?

2. In an $n \times n$ array of numbers, all rows are different (two rows are said to be different if they differ in at least one entry). Prove that there is a column that can be deleted in such a way that the resulting rows will still be different.

(Junior level, 1 point)

3. Consider the $4(N-1)$ squares on the boundary of an $N \times N$ array of squares. We wish to inset in these squares $4(N-1)$ consecutive integers (not necessarily positive) so that the sum of the four vertices of any rectangle with sides parallel to the diagonals of the array (in the case of a "degenerate" rectangle, i.e. a diagonal, we refer to the sum of the numbers in its corner squares) are one and the same number. Is this possible?

   Consider the cases

   (a) $N = 3$,
   (b) $N = 4$, and
   (c) $N = 5$.

(Northern Spring, Junior O paper, 5 points)

4. From a squared piece of paper of size 29×29, 99 pieces, each a 2×2 square, are cut off (all cutting is along the lines bounding the squares). Prove that at least one more piece of size 2×2 may be cut from the remaining sheet.

(Northern Spring, Senior O paper, 5 points)

5. The digits $0, 1, 2, \ldots, 9$ are written in a 10×10 table, each number appearing 10 times.

   (a) Is it possible to write them in such a way that in any row or column there would be not more than 4 different digits?
   (b) Prove that there must be a row or column containing more than 3 different digits.

6. A 7×7 square is made up of 16 3×1 tiles and 1 1×1 tile. Prove that the 1×1 tile lies either at the centre of the square or the adjoins one of its boundaries.

(Northern Autumn, 1984, Senior Main, 10 points)

7. A game has two players. In the game there is a rectangular chocolate bar, with 60 pieces, arranged in a 6×10 formation, which can be broken only along the lines dividing the pieces. The first player breaks the bar along one line, discarding one section. The second player then breaks the remaining section, discarding one section. The first player repeats this process with the remaining section, and so on. The game is won by the player who leaves a single piece. In a perfect game which player wins?