TOURNAMENT OF THE TOWNS, 2004–2005

Training Session, 23 October 2004

JUNIOR QUESTIONS: Years 8, 9, 10

1. In triangle ABC the bisector of angle A intersects the perpendicular bisector of AB at E and the perpendicular bisector of AC at F. Prove that if BE is perpendicular to AC, then CF is perpendicular to AB.

(O Level, Northern Spring 2004, 3 points)

2. Determine all positive integers n for which there exist n consecutive positive integers whose sum is a prime number.

(O Level, Northern Spring 2004, 3 points)

3. From any n-digit number a, we can obtain a $2n$-digit number b by writing two copies of a one after the other. Determine all possible integer values of $\dfrac{b}{a^2}$.

(O Level, Northern Spring 2004, 5 points)

4. The sum of all terms of a finite arithmetic progression of integers is a power of two. Prove that the number of terms is also a power of two.

(A Level, Northern Spring 2004, 4 points)

5. The integer n is the smallest positive multiple of 15 such that every digit of n is either 0 or 2. Compute $n/15$.

(Australian Intermediate Mathematics Olympiad 2000, 2 points)

6. Evaluate the product

$$(\sqrt{2}+\sqrt{11}+\sqrt{13})(\sqrt{2}+\sqrt{11}-\sqrt{13})(\sqrt{2}-\sqrt{11}+\sqrt{13})(-\sqrt{2}+\sqrt{11}+\sqrt{13})$$

(Australian Intermediate Mathematics Olympiad 2000, 2 points)

7. Each of the interior angles of a heptagon (a seven-sided polygon) is obtuse and the number of degrees in each angle is a multiple of 9. No two angles are equal. Find in degrees the sum of the two largest angles in the heptagon.

(Australian Intermediate Mathematics Olympiad 2000, 3 points)

8. Briony takes a standard pack of 52 cards and throws some cards away. However, she makes sure she keeps all four aces among the remaining cards. She then selects four cards at random from these remaining cards. If the probability of her selecting the four aces is $\dfrac{1}{1001}$, how many cards did she throw away?

(Australian Intermediate Mathematics Olympiad 2000, 3 points)

9. A rectangular piece of paper $ADEF$ is folded so that corner D meets the opposite edge EF at D' forming a crease BC where B lies on edge AD and C lies on edge DE. If $|AD| = 25\,\text{cm}$, $|DE| = 16\,\text{cm}$ and $|AB| = 5\,\text{cm}$, find $|BC|^2$.

(Australian Intermediate Mathematics Olympiad 2000, 3 points)

SENIOR QUESTIONS: Years 11, 12

1. Let n and a be positive integers. Prove that the fraction

$$\frac{(a+1)^{2n+1} + a^{n+2}}{a(a+1)+1}$$

is a positive integer.

(AMOC Senior Contest 2000, 7 points)

2. Determine all functions f defined for all real numbers and taking real numbers as their values such that

$$f(x) + xf(1-x) = x^2 - 1$$

holds for each real number x.

(AMOC Senior Contest 2000, 7 points)

3. Prove that

$$\frac{n}{2^1(n-1)} + \frac{n}{2^2(n-2)} + \cdots + \frac{n}{2^{n-2}\cdot 2} + \frac{n}{2^{n-1}\cdot 1} < 4$$

for all integers $n > 1$.

(AMOC Senior Contest 2000, 7 points)