# Western Australian
# Junior Mathematics Olympiad 2016

## Individual Questions **100 minutes**

**General instructions:** *Except possibly for Question 12, each answer in this part is a positive integer less than 1000. No working is needed for Questions 1 to 11. Calculators are* ***not*** *permitted. In general, diagrams are provided to clarify wording only, and are not to scale.*
*Write your answers for Questions 1–11, and solution for Question 12 on the front and back, respectively, of the Answer Sheet provided.*

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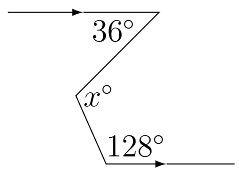
**1.** Find the value of $x$ in the following diagram.




[1 mark]

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**2.** An icecream dessert is made up of a cone and a scoop of icecream such that the scoop of icecream is a perfect hemisphere placed atop the cone.

If the wafer biscuit which makes up the cone is 100 mm along the outside (slant edge) from the top of the wafer to its tip and the radius of the hemisphere is 28 mm, how many millimetres tall is the icecream dessert from the top of the icecream scoop to the tip at the base? [1 mark]

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**3.** I am thinking of two numbers between 1 and 20. The first is one less than a perfect square and is not prime. The second is one more than a perfect square and is prime. The sum of the two numbers is divisible by 4 but not by 8.

What is the product of the two numbers? [1 mark]

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**4.** A jigsaw puzzle is $8 \times 6$, making 48 pieces altogether in a rectangular pattern. Each corner piece is connected to two other pieces, each edge piece which is not on a corner is connected to three other pieces and each of the remaining pieces is connected to four other pieces.

How many connections are there altogether? Be careful not to count the connections twice: if two pieces are connected to each other that's only one connection. [1 mark]

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**5.** A school bus has 45 bench seats. Each bench seat can take two children or one child with their backpack.

If $\frac{2}{3}$ of the children have backpacks, how many children can be seated on the bus? [2 marks]

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**6.** Mr Throttlebottom pays a bill using internet banking. The bill is for a two digit whole number of dollars, say \$$ab$. He accidentally inserts an extra digit after the $b$. Later he is annoyed to discover that he has overpaid the bill by \$647.

How many dollars should he have paid? [2 marks]

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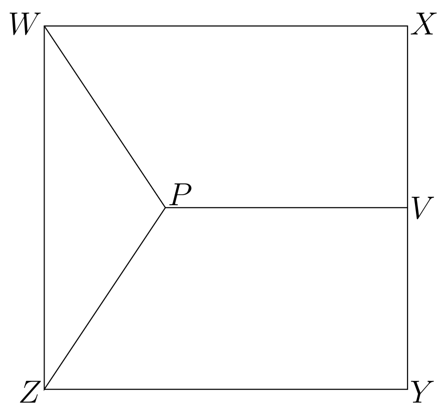
**7.** $WXYZ$ is a square with $PV \perp XY$ and $PW = PZ = PV = 10\,\text{cm}$.

How many $\text{cm}^2$ is the area of $WXYZ$?




[2 marks]

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**8.** How many positive integers less than 300 000 are there that have the digits 5, 6 and 7 next to each other and in that order? [2 marks]

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**9.** In a right-angled triangle $ABC$, a point $M$ on the hypotenuse $BC$ is such that $AM \perp BC$. Also, $MC$ is 8 cm longer than $BM$, and the ratio $AB : AC = 3 : 5$.

How many centimetres is the length of the hypotenuse? [3 marks]

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**10.** In a 5 000 m race, the winner finishes 200 m ahead of second and 600 m ahead of third. Assuming the runners run at a constant speed, to the nearest integer how many metres ahead of third will second finish? [3 marks]

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**11.** Two numbers $x$ and $y$ satisfy three of the following equations

$$x + y = 63, \quad x - y = 47, \quad xy = 392, \quad \frac{x}{y} = 8,$$

but do not satisfy the remaining one.

What is the value of $x$? [3 marks]

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**12.** **For full marks explain how you found your solution.**

Wilberforce walks 1 metre east from his initial position, then 2 metres north, then 3 metres west, then 4 metres south, then 5 metres east, then 6 metres north, and so on, (so that he walks one metre further each time before turning 90° to his left), until he finally walks 41 metres east.

How many metres in a straight line path is Wilberforce's final position from his initial position? [4 marks]

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# Junior Mathematics Olympiad 2016

## Team Question **45 minutes**

**General instructions:** *Calculators are (still)* ***not*** *permitted.*

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**Triangles with Integer Sides**

This problem is about triangles all of whose sides have *integer* length, which we call *integer triangles.*

Remember the **triangle inequality**:

> There exists a triangle with sides of length $a$, $b$, $c$ if and only if each length is less than the sum of the other two.

We say that a triple $(a, b, c)$ of positive integers is a **good triple** if there is an integer triangle with sides of lengths $a$, $b$ and $c$ and $a \leq b \leq c$. The **perimeter** of a good triple $(a, b, c)$ is the perimeter of the triangle with sides $a$, $b$ and $c$, that is $a + b + c$. We call a good triple an **odd triple** if the perimeter is odd, and an **even triple** if the perimeter is even. So $(3, 4, 4)$ is an odd triple and $(2, 2, 2)$ is an even triple.

If $n$ is any positive integer, let $T(n)$ be the number of good triples of perimeter $n$. For example $T(3) = 1$ because the only side lengths of an integer triangle of perimeter 3 are $(1, 1, 1)$ and $T(5) = 1$ because every integer triangle with perimeter 5 is congruent to a triangle with side lengths $1, 2, 2$.

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**A.** Explain why $T(4) = 0$.

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**B.** Show that $T(6) = 1$.

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**C.** Show that if $(a, b, c)$ is a triple of positive integers satisfying $a \leq b \leq c$ and $a + b > c$, then $(a, b, c)$ is a *good* triple.

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**D.** Explain why a good triple $(a, b, c)$ of perimeter $n$ has $a \leq n/3$.

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**E.** Let $(a, b, c)$ be an even triple. Explain why $a \geq 2$.

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**F.** Show that $T(12) = 3$.

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**G.** Fill in the table on the back of the cover page listing all good triples and $T(n)$ for $n = 1, 2, \ldots, 14$.

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**H.** Let $(a, b, c)$ be a good triple with perimeter $n$. Find the largest possible value for $v$ and smallest value for $w$ such that

$$vn \leq c < wn,$$

where $v$ and $w$ are fractions. Explain why this double inequality (with the values you found for $v$ and $w$) holds for all good triples.

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**I.** Let $(a, b, c)$ be an even triple. Explain why $(a-1, b-1, c-1)$ is a good triple which is odd.

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**J.** Let $(a-1, b-1, c-1)$ be an odd triple. Explain why $(a, b, c)$ is a good triple that is even.

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**K.** Explain why if $n$ is even then $T(n) = T(n-3)$.

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## Individual Questions Solutions

**1.** Answer: 88. Drawing in another parallel through the angle with value $x°$, we see that $x°$ is the sum of an alternate angle and a cointerior angle, i.e.

$$x = 36 + (180 - 128) = 88.$$

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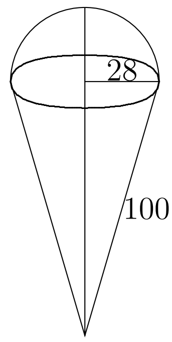
**2.** Answer: 124. The vertical height of the cone is




$$\begin{aligned}\sqrt{100^2 - 28^2} &= \sqrt{4^2(25^2 - 7^2)}\\ &= \sqrt{4^2 \cdot 24^2}, \text{ using the Pythagorean triple: 7, 24, 25}\\ &= 4 \cdot 24\\ &= 96.\end{aligned}$$

Hence the total height is $96 + 28 = 124\,\text{mm}$.

**Alternatively,** the square of the vertical height of the cone is

$$\begin{aligned}100^2 - 28^2 &= (100 - 28)(100 + 28)\\ &= 72 \cdot 128\\ &= 2^3 \cdot 3^2 \cdot 2^7\\ &= (2^5 \cdot 3)^2\\ \therefore\ \sqrt{100^2 - 28^2} &= 2^5 \cdot 3\\ &= 96.\end{aligned}$$

Hence the total height is $96 + 28 = 124\,\text{mm}$.

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**3.** Answer: 75. The first number must be 8 or 15. The second must be 2, 5 or 17. The possible sums are $8 + 2 = 10$, $8 + 5 = 13$, $8 + 17 = 25$, $15 + 2 = 17$, $15 + 5 = 20$ and $15 + 17 = 32$; of these sums, only 20 is divisible by 4 but not by 8.
So the numbers are 15 and 5 and their product is 75.

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**4.** Answer: 82. We have columns of 6 pieces vertically and rows of 8 pieces horizontally. Counting vertical connections, we have 7 columns of 6 connections, giving 42 connections. Counting horizontal connections, we have 5 rows of 8 connections, giving 40 more connections. Hence the total number of connections is $40 + 42 = 82$.
**Generally,** for an $x \times y$ jigsaw we'll get $(x - 1)y + (y - 1)x$ connections.
**Alternatively,** the four corner pieces contribute $4 \times 2 = 8$ connections.
The pieces on the four sides contribute $(4 + 4 + 6 + 6) \cdot 3 = 60$ connections.
There are $4 \times 6 = 24$ other pieces and these will contribute $4 \times 24 = 96$ connections.
Adding these up gives a total of $8 + 60 + 96 = 164$ connections. *However*, in this calculation we've counted each connection twice so we need to divide by 2. So the answer is $164/2 = 82$.

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**5.** Answer: 54. Let the number of children be $x$. Then $2x/3$ have backpacks and $x/3$ are without backpacks. These $x$ children need $2x/3 + x/6 = 5x/6$ bench seats. Since there are 45 seats,

$$\begin{aligned} \frac{5x}{6} &= 45 \\ x &= \tfrac{6}{5} \cdot 45 \\ &= 54. \end{aligned}$$

So 54 children are seated on the bus.

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**6.** Answer: 71. Let the extra digit be denoted by $c$. Then he's paid $\$(100a + 10b + c)$ instead of $\$(10a + b)$, so

$$\begin{aligned} 647 &= (100a + 10b + c) - (10a + b) \\ &= 9(10a + b) + c. \end{aligned}$$

This means that $c$ is the remainder when 647 is divided by 9, so $c = 8$, then

$$10a + b = \frac{647 - 8}{9} = \frac{639}{9} = 71.$$

So he should only have paid \$71.

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**7.** Answer: 256. Let $s$ be the side length of the square. Then we want $s^2$ where

$$\begin{aligned} (s - 10)^2 + (s/2)^2 &= 10^2 \\ \tfrac{5}{4}s^2 - 20s &= 0 \\ \tfrac{5}{4}s &= 20 \\ s^2 &= (\tfrac{4}{5} \cdot 20)^2 \\ &= 16^2 = 256. \end{aligned}$$

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**8.** Answer: 900. The numbers can be of the form $x567yz$, $xy567z$ or $xyz567$, where in each case, $x$ is empty (i.e. 0), or 1 or 2, and each of $y$ and $z$ can be any digit, giving $3 \times 10 \times 10$ choices for $xyz$. Hence there are $3 \times 3 \times 10 \times 10 = 900$ possibilities.

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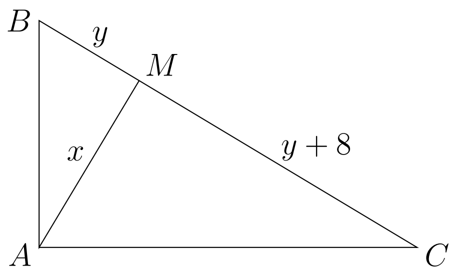
**9.** Answer: 17. Let $x$ be the length of $AM$ and $y$ the length of $BM$. The three triangles $ABC$, $BMA$ and $AMC$ are similar. So

$$\frac{x}{y} = \frac{y+8}{x} = \frac{5}{3},$$

and the length we are after is $BC = 2y+8$. Hence

$$\begin{aligned} x = \frac{5y}{3} \text{ and } y+8 &= \frac{5x}{3} \\ &= \frac{25y}{9} \\ 25y &= 9y+72 \\ 16y &= 72 \\ 2y &= 9 \\ BC &= 2y+8 = 17. \end{aligned}$$

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**10.** Answer: 417. Let $u$, $v$ and $w$ be the speeds of the first, second and third places, and let $t_1$ be the time taken by first place. Then, since

$$\begin{aligned} \text{time} &= \frac{\text{distance}}{\text{speed}} \\ (t_1 =)\ \frac{5000}{u} &= \frac{5000-200}{v} = \frac{5000-600}{w} \\ \therefore\ v &= \frac{4800}{5000}u = \frac{24}{25}u \\ w &= \frac{4400}{5000}u = \frac{22}{25}u \end{aligned}$$

Now let $d$ be the distance between second and third at the time $t_2$ when second place crosses the finish line. Then

$$\begin{aligned} (t_2 =)\ \frac{5000}{v} &= \frac{5000-d}{w} \\ 5000 \cdot \frac{25}{24} \cdot \frac{1}{u} &= (5000-d) \cdot \frac{25}{22} \cdot \frac{1}{u} \\ 5000 \cdot \frac{22}{24} &= 5000-d \\ d = 5000\left(1-\frac{22}{24}\right) &= 5000 \cdot \frac{2}{24} = \frac{1250}{3} = 416\tfrac{2}{3}\,\text{m}, \end{aligned}$$

which rounds to 417 m.

**Alternatively,** let $t_1$, $u$, $v$ and $w$ be as above. Then, as before

$$\begin{aligned} t_1 &= \frac{5000}{u} = \frac{4800}{v} = \frac{4400}{w} \\ \therefore\ v &= \frac{4800}{t_1},\ w = \frac{4400}{t_1}. \end{aligned}$$

To run 200 m, the second runner needs time

$$\frac{200}{v} = 200 \cdot \frac{t_1}{4800} = \frac{t_1}{24}.$$

In that time, the third runner will run a distance of

$$w \cdot \frac{t_1}{24} = \frac{4400}{t_1} \cdot \frac{t_1}{24} = \frac{4400}{24} = \frac{550}{3} = 183\tfrac{1}{3}\,\mathrm{m},$$

so that in the time the second runner runs the last $200\,\mathrm{m}$, the third runner runs $183\frac{1}{3}\,\mathrm{m}$. Before running those respective final distances the second and third runners were $400\,\mathrm{m}$ apart. So when the second runner crosses the finish line, the second and third runners are

$$400 + 200 - 183\tfrac{1}{3} = 416\tfrac{2}{3} \approx 417\,\mathrm{m}$$

apart.

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**11.** Answer: 56. Listing the equations in order to label them:

$$\begin{aligned} x + y &= 63 && (1)\\ x - y &= 47 && (2)\\ xy &= 392 && (3)\\ \frac{x}{y} &= 8 && (4) \end{aligned}$$

Suppose both (1) and (2) are satisfied. Then

$$\begin{aligned} (1)+(2):\quad 2x &= 110\\ x &= 55\\ (1):\quad y &= 63 - 55 = 8. \end{aligned}$$

But then $xy = 440 \neq 392$ and $x/y = 55/8 \neq 8$, so that neither (3) nor (4) is satisfied. So one of (1) and (2) is not satisfied, but both of (3) and (4) must be satisfied. Hence,

$$\begin{aligned} (3)\cdot(4):\quad x^2 &= 392 \cdot 8\\ &= 14^2 \cdot 2 \cdot 8\\ &= (14 \cdot 4)^2\\ x &= \pm 56 \end{aligned}$$

If $x = -56$ then by (4), $y = -7$, so that

$$x + y = -63 \text{ and } x - y = -49,$$

meaning neither (1) nor (2) is satisfied. Hence $x = 56$, and so by (4), $y = 7$ and (1) is satisfied, but not (2).

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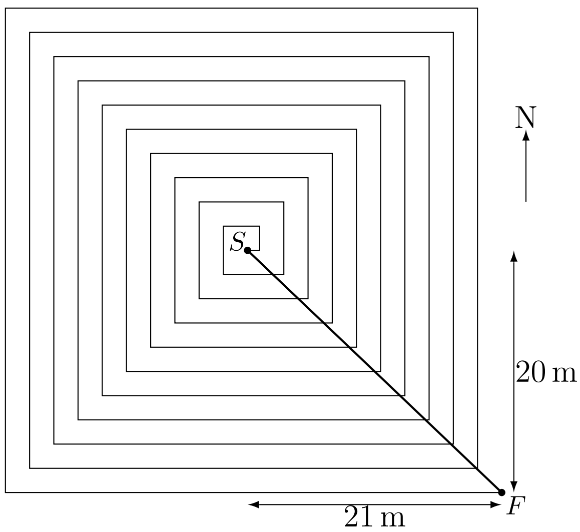
**12.** Answer: 29. The net distance Wilberforce walks eastwards is

$$\begin{aligned} &1-3+5-7+\cdots+41 \\ &\quad=(1-3)+(5-7)+\cdots+(37-39)+41 \\ &\quad=10\times(-2)+41 \\ &\quad=21\,\mathrm{m} \end{aligned}$$




The net distance Wilberforce walks northwards is

$$\begin{aligned} &2-4+6-8+\cdots+38-40 \\ &\quad=(2-4)+(6-8)+\cdots+(38-40) \\ &\quad=10\times(-2) \\ &\quad=-20\,\mathrm{m} \end{aligned}$$

So he finishes 20 m south of his original position. His distance from his initial position is therefore, using Pythagoras' Theorem,

$$\sqrt{21^2+(-20)^2}=\sqrt{441+400}=\sqrt{841}=29.$$

**Remark.** Here we have an integer Pythagorean triple $(a,b,c)$ with $b=a+1$, so that

$$\begin{aligned} a^2+(a+1)^2&=c^2 \\ 2a^2+2a+1&=c^2 \\ 2a(a+1)&=c^2-1 \\ &=(c-1)(c+1) \end{aligned}$$

Since the left hand side of the last equation is even, so is the right hand side, and hence at least one of the factors $c-1$ and $c+1$ is even. But $c-1$ and $c+1$ differ by 2, and so must be of the same parity, and hence are both even. Looking again at the lefthand side, we see $a$ and $a+1$ are of opposite parity. So one is odd and the other even. So the left hand side is in fact divisible by 4. Thus the last equation reduces to:

$$\frac{a(a+1)}{2}=\frac{c-1}{2}\cdot\frac{c+1}{2},$$

where the left hand side is a triangular number and the right hand side is the product of consecutive integers. For the numbers in our problem, we have

$$T_{20}=\frac{20\cdot 21}{2}=14\cdot 15,\ \text{and } c=29.$$

So a natural question to ask is:

> How frequently does an integer Pythagorean triple of the form $(a,a+1,c)$ happen?

The answer is that it's rare. The following are the triples for $c<10\,000$:

$$(3,4,5),\ (20,21,29),\ (119,120,169),\ (696,697,985),\ (4059,4060,5741).$$

**Triangles with Integer Sides**

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**A.** To have integer sides that sum to 4, a triangle $(a, b, c)$ would have to have sides of lengths 1 or 2 and no such triangle has perimeter 4. That is, the only potential triple satisfying $a \le b \le c$ is $(1, 1, 2)$ which violates the triangle inequality, since $1 + 1 = 2 \not> 2$.
So $T(4) = 0$.

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**B.** Systematically listing integer partitions of 6 with $a \le b \le c$, we have

$$(1, 1, 4),\ (1, 2, 3),\ (2, 2, 2).$$

Of these only $(2, 2, 2)$ satisfies the *triangle inequality*. $\therefore T(6) = 1$.
**Alternatively,** if $c \ge 3$ then since $a + b > c$ we have $a + b + c > 6$, a contradiction. Hence, $c$ is at most 2, but now if any of $a$, $b$ or $c < 2$ then $a+b+c < 6$, a contradiction. Hence, $a = b = c = 2$ and so $(2, 2, 2)$ is the only good triple for perimeter 6.
$\therefore T(6) = 1$.

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**C.** In order to show that a triple with $a \le b \le c$ is good, we need to show the *triangle inequality* holds, i.e.

$$\begin{cases} a < b + c & (1) \\ b < a + c & (2) \\ c < a + b & (3) \end{cases}$$

Inequality (3) is assumed. However, the condition $a \le b \le c$ with $a, b, c$ positive implies

$$b \le c < c + a,$$

so that (2) is automatically satisfied. Similarly,

$$a \le b < b + c,$$

so that (1) is automatically satisfied.

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**D.** Suppose for a contradiction that $a > n/3$. Then since $a$ is the smallest side, $b > n/3$ and $c > n/3$. Hence

$$a + b + c > n/3 + n/3 + n/3 = n,$$

i.e. the perimeter is larger than $n$, a contradiction.
$\therefore a \le n/3$.

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**E.** Suppose $(a, b, c)$ is an even triple with $a = 1$. Since $1 + b + c$ is even, $b$ and $c$ have opposite parity, so $c - b \ge 1$ and hence $1 + b \le c$, contradicting the triangle inequality. Hence $a \ge 2$.

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**F.** By **D.** and **E.** we have $2 \le a \le 4$. With this restriction on $a$, systematically listing integer partitions of 12 with $a \le b \le c$, we have

$$\begin{cases} (2, 2, 8), (2, 3, 7), (2, 4, 6), (2, 5, 5), \\ (3, 3, 6), (3, 4, 5), \\ (4, 4, 4). \end{cases}$$

Of these only $(2, 5, 5)$, $(3, 4, 5)$, and $(4, 4, 4)$ satisfy the *triangle inequality*.
$\therefore T(12) = 3$.

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**G.**

| $n$ | $T(n)$ | Triples |
|---|---|---|
| 1 | 0 | — |
| 2 | 0 | — |
| 3 | 1 | $(1, 1, 1)$ |
| 4 | 0 | — |
| 5 | 1 | $(1, 2, 2)$ |
| 6 | 1 | $(2, 2, 2)$ |
| 7 | 2 | $(1, 3, 3)$, $(2, 2, 3)$ |
| 8 | 1 | $(2, 3, 3)$ |
| 9 | 3 | $(1, 4, 4)$, $(2, 3, 4)$, $(3, 3, 3)$ |
| 10 | 2 | $(2, 4, 4)$, $(3, 3, 4)$ |
| 11 | 4 | $(1, 5, 5)$, $(2, 4, 5)$, $(3, 3, 5)$, $(3, 4, 4)$ |
| 12 | 3 | $(2, 5, 5)$, $(3, 4, 5)$, $(4, 4, 4)$ |
| 13 | 5 | $(1, 6, 6)$, $(2, 5, 6)$, $(3, 4, 6)$, $(3, 5, 5)$, $(4, 4, 5)$ |
| 14 | 4 | $(2, 6, 6)$, $(3, 5, 6)$, $(4, 4, 6)$, $(4, 5, 5)$ |

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**H.** Answer: $n/3 \le c < n/2$, i.e. $v = \frac{1}{3}$ and $w = \frac{1}{2}$.
Checking the table in **G.**, we see for instance that for $n = 9$, $c$ can be 3 or 4, for $n = 12$, $c$ can be 4 or 5, and for $n = 7$, $c$ can be only 3. From these data we guess the above values for $v$ and $w$.
Suppose for a contradiction that $c \ge n/2$. Then since $a + b > c$, we have

$$a + b + c > 2c \ge n,$$

i.e. the perimeter is larger than $n$, a contradiction.
$\therefore c < n/2$, i.e. $w = \frac{1}{2}$.
Now suppose $c < n/3$. Then since $c$ is the largest side we have $a < n/3$ and $b < n/3$ also, and so

$$a + b + c < n/3 + n/3 + n/3 = n,$$

i.e. the perimeter is less than $n$, again a contradiction.
$\therefore c \ge n/3$, i.e. $v = \frac{1}{3}$. Note that the example $(2, 2, 2)$ shows that we cannot do better than $v = \frac{1}{3}$.
$\therefore n/3 \le c < n/2$.

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**I.** By **C.**, we need only show $a-1 \le b-1 \le c-1$ and $c-1 < (a-1)+(b-1)$. The first statement follows from $a \le b \le c$. Let us now prove the second inequality. Since $a+b$ and $c$ sum to an even number, $a + b$ and $c$ have the same parity. Moreover, $c < a + b$ and so $a+b \ge c+2$. Hence $a-1+b-1 \ge c > c-1$, and so $(a-1, b-1, c-1)$ is a good triple which is odd because with $a+b+c$ even, $(a-1)+(b-1)+(c-1) = a+b+c-3$ is necessarily odd.

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**J.** By **C.**, we need only show $a \leq b \leq c$ and $c < a + b$. The first statement follows from $a-1 \leq b-1 \leq c-1$. Let us now prove the second inequality. If $(a-1, b-1, c-1)$ is good then $a+b-2 > c-1$ so that $a+b > c+1 > c$, and since it is an odd triple, $a+b+c-3$ is odd and hence $a+b+c$ is even. Thus $(a, b, c)$ is a good triple which is even.

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**K.** First we can check in table **G.** that the statement is true for $n = 4, 6, 8, 10, 12, 14$. Since we only defined $T(n)$ for positive integers $n$, we need $n \geq 4$ in order that the right hand side of $T(n) = T(n-3)$ be defined.

By **I.**, if $n$ is even, for every good triple $(a, b, c)$ for perimeter $n$ there is a good triple $(a-1, b-1, c-1)$ of perimeter $n-3$. Moreover, distinct triples $(a, b, c)$ yield distinct triples $(a-1, b-1, c-1)$. Thus we have $T(n) \leq T(n-3)$.

If $n$ is even, then $n-3$ is odd, and so by **J.**, for every good triple $(a-1, b-1, c-1)$ of perimeter $n-3$, there is a good triple $(a, b, c)$ of perimeter $n$. Moreover, distinct triples $(a-1, b-1, c-1)$ yield distinct triples $(a, b, c)$. Thus we have $T(n-3) \leq T(n)$.

Since we have both $T(n) \leq T(n-3)$ and $T(n-3) \leq T(n)$, we have

$$T(n) = T(n-3), \text{ if } n \text{ is even.}$$

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