CHAPTER 5

Arithmetic sequences

5.1 Sequences

A **sequence** is an *ordered* list of numbers usually represented as numbers separated by commas:

$$a_1, a_2, \ldots, a_n, \ldots.$$

Each such number of the sequence is called a **term**, with a_n being the n^{th} *term*, and the count n is the **index** of the term.
There are two main ways to define a sequence, by:

recurrence. In this case, a_n is a function of previous terms,

$$a_n = f(a_{n-1}, \ldots, a_{n-k}).$$

The difference of the highest and lowest index (here: $n - (n - k) = k$) that appears in the *recurrence*, is its **order**, and is the number of *initial terms* whose values must be given to fully determine the sequence.
For example, the **Fibonacci sequence**,

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

(with n^{th} term F_n) is defined by

$$\begin{aligned} &F_n = F_{n-1} + F_{n-2},\ n \geqslant 3. \\ &F_1 = F_2 = 1, \end{aligned}$$

where the first equation, is its defining *recurrence.* Observe that the *recurrence* has *order* 2, and so it is accompanied with the definition of its first 2 terms.

explicit rule. In this case, a_n is defined by some function of n,

$$a_n = f(n).$$

For example, the **square sequence**,

$$1, 4, 9, 16, \ldots$$

is definied by

$$a_n = n^2,\ n \geqslant 1.$$

5.2 Arithmetic sequence

An **arithmetic sequence** is defined by the *recurrence*

$$\begin{aligned} &a_n = a_{n-1} + d,\ n \geqslant 2, \\ &a_1 = a \end{aligned}$$

where d is called the **common difference**.

The recurrence says that each term is obtained from the previous term by adding the *common difference* d, or put another way, consecutive terms of the the sequence always differ by the same constant d.

Since the recurrence has *order* 1, one term (here: a) must be given.

Observe that in writing the terms a_1 up to a_n in the usual way, the n terms are separated by $n-1$ commas. For an *arithmetic sequence*, these commas count the number of differences we need to add to $a_1 = a$ in order to obtain a_n; so we have the *explicit rule*,

$$a_n = a + (n-1)d, \; n \geqslant 1,$$

for an *arithmetic sequence*, for constants a and d.

Example 5.2.1. *The sequence of odd natural numbers:* $1, 3, 5, 7, \ldots$ *is arithmetic since*

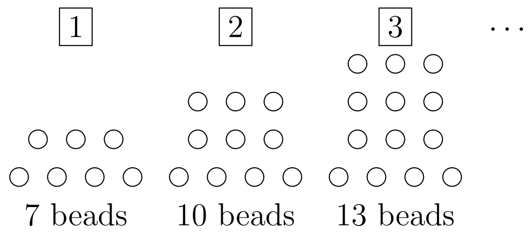
$$3 - 1 = 5 - 3 = 7 - 5 = \cdots = 2,$$

i.e. it has a common difference $d = 2$*. Also, the sequence's first term* $a = 1$*.*

Non-example. The sequence of *powers of 2*: $1, 2, 4, 8, \ldots$ is not *arithmetic* since, considering the first couple of term differences,

$$2 - 1 = 1 \neq 2 = 4 - 2.$$

Example 5.2.2. *Each of the diagrams below represent rows of beads.*



The sequence formed: $7, 10, 13, \ldots$ *is arithmetic since the common difference* $d = 3$*.*
Also, the sequence's first term $a = 7$*.*
How many beads will there be in the seventh diagram?

Solution. The number of beads in the 7^{th} diagram, is the 7^{th} term a_7 and so, via the *explicit rule*,

$$\begin{aligned} a_7 &= 7 + (7-1)3 \\ &= 25. \end{aligned}$$

So the 7^{th} diagram will have 25 beads. □

Example 5.2.3. *The multiples of* 3 *form an arithmetic sequence.*
We can see directly that its explicit rule is: $a_n = 3n$,
and both the first term a *and the common difference* d *is* 3*.*

Example 5.2.4. *Find the 17^{th} and the n^{th} term of the arithmetic sequence,*

$$4, 7, 10, \ldots$$

Solution. Observe that the sequence's *first term* $a = 4$, and its *common difference* $d = 3$. Thus the 17^{th} term is

$$\begin{aligned} a_{17} &= 4 + (17 - 1)3 \\ &= 52, \end{aligned}$$

and the n^{th} term is

$$\begin{aligned} a_n &= 4 + (n - 1)3 \\ &= 4 + 3n - 3 \\ &= 3n + 1. \end{aligned}$$

□

5.3 The sum of n terms of an arithmetic sequence

There is a famous story about Carl Friedrich Gauss while he was still in primary school. Gauss' teacher asked his class to total all the integers from 1 to 100, presuming that this task would occupy the students for quite a while. However, the 8-year-old Gauss, returned the correct answer 5050 after only a few seconds. Essentially what Gauss did was to write the sum – call it S_{100} – (though he did it in head) wrapping it around:

$$\begin{aligned} S_{100} = \quad 1 + \ \ 2 + \ \ 3 + \cdots + 49 + 50 +& \\ 100 + 99 + 98 + \cdots + 52 + 51& \end{aligned}$$

Doing so, Gauss had 50 columns of pairs of numbers, where each pair summed to 101, and hence $S_{100} = 5050$. That trick works well when the number of terms n is even, but not quite so well when n is odd. Tweaking the idea, we write the numbers forward on one line and in reverse on the next line. So with n numbers in the sum, first term a, common difference d and last ($= n^{\text{th}}$) term ℓ, we have:

$$\begin{aligned} S_n &= a + (a + d) + (a + 2d) + \cdots + \ell \\ &= \ell + (\ell - d) + (\ell - 2d) + \cdots + a \\ 2S_n &= n(a + \ell) \\ S_n &= \frac{n(a + \ell)}{2} \end{aligned}$$

and, if we don't know ℓ in advance, from the *explicit rule*,

$$\begin{aligned} \ell &= a + (n - 1)d \\ S_n &= \frac{n}{2}\big(a + a + (n - 1)d\big) \\ &= \frac{n}{2}\big(2a + (n - 1)d\big). \end{aligned}$$

What's particularly interesting is that (in the case, we know the last term ℓ), the formula $S_n = n(a + \ell)/2$ doesn't depend on d.

Example 5.3.1. *Find the sum of the first 16 multiples of 5.*

Solution. The sequence of multiples of 5: $5, 10, \dots, 5n, \dots$ is *arithmetic* with *first term* $a = 5$, *common difference* $d = 5$ and n^{th} *term* $a_n = 5n$.
The sum to n terms S_n has *last term* $\ell = a_n = 5n$. Hence,

$$\begin{aligned} S_n &= \frac{n(5+5n)}{2} \\ &= \frac{5n(n+1)}{2} \\ \therefore\ S_{16} &= \frac{5 \cdot 16(16+1)}{2} \\ &= 680. \end{aligned}$$

□

Initial problem. The King of Nanastam sent out invitations for the reception for his daughter's marriage. When the guests arrived they did so in an unusual way:

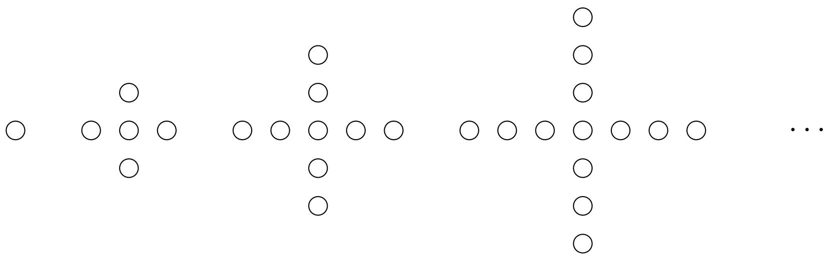
The first group that arrived was a party of 3 people.
The second group of arrivals was a party of 8 people.
The third group was a party of 13 people.

The king noted that each arriving party had 5 more than the previous party.
How many guests arrived for the reception once the 50^{th} and last party had arrived?

Exercise Set 5.

1. Find the 27^{th} term of the arithmetic sequence: $3, 11, 19, \dots$.
2. Find the sum of the first 20 terms of the arithmetic sequence: $3, 11, 19, \dots$.
3. Find the sum of the first
   (a) 5 odd numbers. (b) 20 odd numbers. (c) N odd numbers.
4. Find the
   (a) 20^{th} odd number. (b) N^{th} odd number. (c) M^{th} odd number.
   (d) Show that the sum of the N^{th} odd number and the M^{th} odd number is one less that the $(N+M)^{\text{th}}$ odd number.
   Hence show the sum of two odd numbers is even.
5. Find the
   (a) 20^{th} even number. (b) N^{th} even number.
   (c) Show that the sum of the N^{th} even number and the M^{th} even number is the $(N+M)^{\text{th}}$ even number.
   Hence show the sum of two even numbers is even.

6. The following patterns of marbles



correspond to the arithmetic sequence: $1, 5, 9, 13, \ldots$.

 (a) Find the number of marbles in the 10^{th} pattern of marbles.
 (b) How many marbles are there in the first 15 marble patterns?

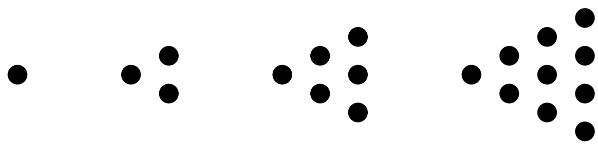
7. Find the sum of the first n natural numbers.

8. The symbol for the dollar is $ or $.
For $, the stroke partitions the S into 4 parts.
For $, the strokes partition the S into 7 parts.

 (a) Suppose the symbol could be drawn with 100 vertical strokes.
 Into how many parts would the S be partitioned?
 (b) If there are N strokes, how many parts will there be?

9. Find 3 integers in arithmetic sequence whose product is prime.

10. The numbers $1, 3, 6, 10$ are the first 4 *triangular numbers.*
These numbers correspond to the following dot configurations.



1 3 6 10

 (a) Find the 15^{th} term of this sequence, i.e. find the 15^{th} triangular number.
 (b) Find the n^{th} triangular number.

11. The interior angles of a convex polygon form an arithmetic sequence.
The smallest angle is $120°$, and the common difference is $5°$.
Find the number of sides of the polygon.

12. At a party, the host's doorbell rang 20 times.
The first time the doorbell rang, one guest arrived.
Each subsequent ringing of the bell, 2 more guests arrived than on the previous ring.
How many guests came to the party?

13. Given a set of n points in the plane, no 3 of which are collinear (i.e. no 3 points are on the same line), how many line segments can be formed?
E.g. with 4 points, 6 line segments can be formed.