

Least common multiple

3.1 Least common multiple

For $a, b \in \mathbb{N}$, a *common multiple* is a number that is *both* a multiple of a and of b , and the **least common multiple** of a and b , written $\text{lcm}(a, b)$, is the *smallest* natural number that is a common multiple of a and b , e.g.

$$\begin{aligned}\text{lcm}(2, 3) &= 6 \\ \text{lcm}(3, 4) &= 12.\end{aligned}$$

The definition of lcm can be extended to more than 2 numbers: the $\text{lcm}(a_1, a_2, \dots, a_n)$ is the least natural number that is a multiple of each of a_1, a_2, \dots, a_n .

One method for finding an lcm is to:

- list the prime factorisations of each of the operand numbers,
- identify the highest power $p_i^{e_i}$ of each prime p_i that appears in these factorisations,
- and
- form the product of all such $p_i^{e_i}$.

Example 3.1.1. Find $\text{lcm}(1080, 25200)$.

Solution.

$$\begin{aligned}1080 &= 2^3 \times 3^3 \times 5 \\ 25200 &= 2^4 \times 3^2 \times 5^2 \times 7 \\ \therefore \text{lcm}(1080, 25200) &= 2^4 \times 3^3 \times 5^2 \times 7 \\ &= 75600.\end{aligned}$$

□

Example 3.1.2. Find the least common multiple of the first ten natural numbers.

Solution.

$$\begin{aligned}2 &= 2^1 \\ 3 &= 3^1 \\ 4 &= 2^2 \\ 5 &= 5^1 \\ 6 &= 2^1 \times 3^1 \\ 7 &= 7^1 \\ 8 &= 2^3 \\ 9 &= 3^2 \\ 10 &= 2^1 \times 5^1 \\ \therefore \text{lcm}(1, 2, \dots, 10) &= 2^3 \times 3^2 \times 5^1 \times 7^1 \\ &= 2520.\end{aligned}$$

□

Example 3.1.3. Find the smallest $n \in \mathbb{N}$ which

when divided by 3, the remainder is 1;
 when divided by 4, the remainder is 2;
 when divided by 5, the remainder is 3; and
 when divided by 6, the remainder is 4.

Solution. Observe that

$$1 = 3 - 2$$

$$2 = 4 - 2$$

$$3 = 5 - 2$$

$$4 = 6 - 2,$$

so that in each case the remainder is 2 less than the divisor.

Thus $n + 2$ is a multiple of each of 3, 4, 5 and 6. Hence,

$$\begin{aligned} n &= \text{lcm}(3, 4, 5) - 2 \\ &= 3 \times 2^2 - 2 \\ &= 60 - 2 \\ &= 58. \end{aligned}$$

□

Initial problem. A girl had amassed a coin collection of 1 c pieces, and observed the following:

division of the number of coins by each of 2, 3, 4, 5, or 6, left a remainder of 1 c,
 the number of coins was divisible by 7, and
 there were between 700 and 800 coins.

What was the amount the girl had collected?

Exercise Set 3.

1. Find

(a) $\text{lcm}(8, 12, 16)$

(b) $\text{lcm}(20, 30, 35)$

(c) $\text{lcm}(6, 8, 24, 30)$

2. Find

(a) $\text{lcm}(6, 7, \dots, 10)$

(c) $\text{lcm}(1, 2, \dots, 20)$

(b) $\text{lcm}(11, 12, \dots, 20)$

(d) $\text{lcm}(585, 10985)$

3. Find the smallest natural number with the property that on division by each of 3, 4 or 5, the remainder is 1.

4. Four ships left a port at noon on 2 January, 1993.
The ships return to the port every 4 weeks, 8 weeks, 12 weeks, and 16 weeks, respectively.
When did all 4 ships next dock together at the port?
5. A 3-digit number has the property that
 - after subtracting 7, the resulting number is divisible by 7,
 - after subtracting 8, the resulting number is divisible by 8, and
 - after subtracting 9, the resulting number is divisible by 9.

What is the number?

6. How many spots are there on a dalmatian, given
 - the number of spots is divisible by 3,
 - dividing the number of spots by the number of legs, yields a remainder of 3,
 - and
 - dividing the number of spots by the total number of legs, ears, eyes and tail, leaves a remainder of 6.
7. Three ships leave New York for France on the same day.
Round trips for the ships take 12 days, 16 days and 20 days, respectively.
Assuming continuous operation, how many days elapse before the ships again leave New York together?
8. In a cafeteria, the timers for quiche and pies buzz every 14 minutes and every 6 minutes, respectively.
If the timers just buzzed together, when next will the timers buzz together?
9. I am thinking of a number. The least common multiple of my number and 9 is 45.
What could my number be?
10. Two neon signs were turned on at the same time.
If the signs blink every 4 seconds, and every 6 seconds, respectively,
how many times per minute do the signs blink together?
11. The Scotch College and Brighton Grammar School cafeterias repeat their lunch menus every 16 days, and every 9 days, respectively.
Both schools served pizza today. When next will both schools serve pizza?