The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

1995 Australian Intermediate Contest Problems

1. Consider the following infinite sequence of digits.

$$1234567891011121314 15\ldots 96979899100101102103\ldots$$

Note that it is formed by writing the base ten natural numbers in order.

Find the $1\,000\,000^{\text{th}}$ digit in this sequence.

2. A computer is programmed so that on input of two base ten numbers, it prints out the two numbers, their sum and their product, in base ten and base two.

George entered two numbers and obtained a print-out. He got the paper wet on the way home and could not tell what most of the digits were. To make it worse, he forgot the numbers he used for input. The print-out looked like this where $*$ means a digit was present but unreadable.

NUMBERS (BASE TEN):	$X = **,\quad Y = **$
NUMBERS (BASE TWO):	$X = *0***,\ Y = ****$
SUM (BASE TEN):	$X + Y = **$
SUM (BASE TWO):	$X + Y = *****1$
PRODUCT (BASE TEN):	$XY = 3**$
PRODUCT (BASE TWO):	$XY = ******1**$

What were the base ten numbers, George put into the computer?

3. If $n \in \mathbb{N}$, prove that
$$(n+1)^n \geq 2^n n!.$$
Under what conditions, does equality occur?

4. Two parallel lines are tangents to a circle with centre O, and a third line also tangent to the circle, meets the two parallel lines at A and B.

Prove that $\angle AOB$ is a right angle.

5. Let $\triangle ABC$ have area 1, let x be such that $0 < x \leq 1$, and let A', B', C' be points on BC, CA, AB, respectively, such that $BA' : A'C = CB' : B'A = AC' = (1-x) : x$.

Find the area of $\triangle ABC$ in terms of x.