The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2001 Australian Intermediate Mathematics Olympiad Problems

1. When a 3-digit number is multiplied by 3 and 1 added, the result is the reverse of the number.

   What is the original number?

2. Let $a, b \in \mathbb{R}$, with $a > 1$ and $b \neq 0$.

   If $ab = a^b$ and $\frac{a}{b} = a^{3b}$, find b^{-a}.

3. The telephone numbers in a large office all have three digits, running from 000 to 999, but not all are in use. If any two digits of a functioning number are interchanged, the resulting number either remains the same or becomes one that is not in use.

   What is the maximum number of functioning numbers possible?

4. Let $\triangle ABC$ be acute-angled with $AB = 15$ and $AC = 13$. Let D be the foot of the perpendicular from A to BC and suppose the area of $\triangle ADC$ is 30.

   Given the area of $\triangle ABC$ is an integer, what is it?

5. Find the number of ordered pairs of integers (x, y) which satisfy

   $$x^2 + 2001 = y^2.$$

6. Find the smallest positive integer n such that there exists an integer m satisfying

   $$0.33 < \frac{m}{n} < \frac{1}{3}.$$

7. In $\triangle ABC$, $\angle BAC = 22°$. A circle with centre O, has AB produced, AC produced and BC as tangents.

   Find the number of degrees of $\angle BOC$.

8. Mr. Bean outsourced one of his soy plots, sized $10\,800\,\mathrm{m}^2$, for processing to three companies specialised in insect treatment: Mount Rid, Legionnaire Futures and Ross River Limited.

   Mount Rid was able to offer a process that would take 3 hours to complete, while Legionnaire Futures quoted 4 hours and Ross River had an offer of 6 hours.

   The three companies agreed that they should all work together, each at their processing capabilities, to complete the treatment. Mr. Bean agreed with this plan and work commenced. When half the soy plot had been treated, Legionnaire Futures had to withdraw. The other two companies sticking to the agreement, together completed the work.

   Find the total number of minutes taken to complete the treatment.

9. A circle, with centre O, has diameter AB. Let C be a point on the circle different from A and B, D be the point on AB such that $\angle CDB = 90°$, and M be the point on BC such that $\angle BMO = 90°$.

   If $DB = 3 \times OM$, calculate $\angle ABC$.

10. The n^{th} triangular number is given by

$$t_n = 1 + 2 + \cdots + n.$$

So the first 12 triangular numbers are:

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78.$$

Notice that 16 can be written as the sum of two triangular numbers in two different ways:

$$t_1 + t_5 = 1 + 15 = 16 = 6 + 10 = t_3 + t_4.$$

The integer 16 is the smallest integer with this property.

(a) (i) Find the next smallest integer that can be written as the sum of two triangular numbers in two different ways.

(ii) Given that $t_{k(n)} = t_n + t_{2n+3} - t_{n+2}$, find a formula for $k(n)$.
Hence show that there are infinitely many integers that are the sum of two triangular numbers in at least two different ways.

(b) Notice that

$$t_1 + t_9 = 1 + 45 = 46 = 10 + 36 = t_4 + t_8.$$

Find a general formula for which this is the first case and use this to find another infinite set of integers, each element of which is the sum of two triangular numbers in at least two different ways.

Investigation.
Are there any integers which are the product of two triangular numbers in two different ways? Are there infinitely many? Investigate.