The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2002 Australian Intermediate Mathematics Olympiad Problems with Solution to Problem 9.

1. The number 888888 is written as the product of two 3-digit numbers.

   Find the larger.

2. $ABCD$ is a trapezium for which $AB \parallel DC$, $AB = 84$ and $DC = 25$. A circle can be drawn in the trapezium so that it just touches all four sides.

   Find the perimeter of the trapezium.

3. Start with three consecutive positive integers. Leave the first unchanged, add 10 to the second and add a prime to the third. The three numbers are now in geometric progression, i.e. they are of the form a, ar, ar^2.

   What was the prime number added to the third of the consecutive integers?

4. In 3-D space, points P and Q, 136 metres apart, are on the same side of a wall and are the same horizontal distance from it. At P a sound signal is generated. When travelling directly, it arrives at Q one tenth of a second earlier than after being reflected off the wall.

   Assuming that sound travels at 340 metres per second and that the angles made by the sound beam and its reflection, with the wall, are equal, determine the distance, in metres, of the point P from the wall.

5. The currency of a small island republic has three different kinds of coins each worth a different integral amount of dollars. Heather received four coins of the republic worth \$28, while David collected five with a total value of \$21. Each had at least one coin of each kind.

   Find, in dollars, the total worth of the three coins.

6. In $\triangle ABC$, $AB = 360$, $BC = 240$ and $AC = 180$. The internal and external angle bisectors of $\angle CAB$ meet BC and BC produced, at P and Q, respectively.

   Find the radius of the circle which passes through A, P and Q.

7. A mathematics student read a telephone number $abc–defg$ and thought that it was a normal base-10 subtraction problem. The answer came out to be -95. Given that the digits of the telephone number were distinct, determine the smallest possible 7-digit number with this property.

   What then is abc?

8. Sarah constructs a large cube from a pile of unit cubes. She then paints some (at least one) of the faces of her large cube blue, then some (at least one) of the remaining faces yellow, and then all the remaining faces red.

Later on, Brett comes along and pulls the large cube apart and discards the unit cubes with no paint on them. Brett takes the remaining cubes (those with some paint on them) and separates them into three piles as follows:

He puts all the unit cubes with some blue on them into pile 1.
Then, he puts the remaining cubes with some yellow on them into pile 2.
Finally, all the units cubes that are left are put into pile 3.

Brett is surprised to find that the numbers of unit cubes in piles 1 and 3 are the same.

Brett puts the three piles together again and forms three new piles as follows:

He puts all the unit cubes with some red on them into pile 1.
Then, he puts the remaining cubes with some yellow on them into pile 2.
Finally, all the units cubes that are left are put into pile 3.

This time Brett finds there are more unit cubes in pile 1 than pile 3, but pile 2 contains the most.

Brett again recombines the three piles and forms three new piles as follows:

He puts all the unit cubes with some blue on them into pile 1.
Then, he puts the remaining cubes with some red on them into pile 2.
Finally, all the units cubes that are left are put into pile 3.

There are now more unit cubes in pile 2 than pile 1, but pile 3 contains the most.

How many unit cubes have some yellow on them?

9. Note that

(1) $$(3n-1)^2+(3n)^2+(3n+1)^2=(5n)^2+(n-1)^2+(n+1)^2;$$
(2) $$(3n)^2+(3n+1)^2+(3n+2)^2=(5n+2)^2+(n-1)^2+n^2.$$

(a) Find three positive integers, which are not consecutive, the sum of whose squares is equal to:

(i) $12^2+13^2+14^2$;

Solution. Substituting $n=4$ in (2):

$$12^2+13^2+14^2=22^2+3^2+4^2.$$

(ii) $11^2+12^2+13^2$;

Solution. Substituting $n=4$ in (1):

$$11^2+12^2+13^2=20^2+3^2+5^2.$$

(iii) $10^2+11^2+12^2$.

Solution. Finding a formula for $(3n + 1)^2 + (3n + 2)^2 + (3n + 3)^2$ (and then substituting $n = 3$) will enable us to also solve (b):

$$\begin{aligned}
(3n+1)^2 + (3n+2)^2 + (3n+3)^2 &= 27n^2 + 2\cdot 3\cdot(1+2+3)n + 1^2 + 2^2 + 3^2\\
&= 27n^2 + 36n + 14\\
&= 25n^2 + 2\cdot 5\cdot 3n + 3^2 + 2n^2 + 6n + 5\\
&= (5n+3)^2 + n^2 + 4n + 4 + n^2 + 2n + 1\\
&= (5n+3)^2 + (n+2)^2 + (n+1)^2\\
\therefore 10^2 + 11^2 + 12^2 &= 18^2 + 5^2 + 4^2.
\end{aligned}$$

Note that this solution is not unique:

$$10^2 + 11^2 + 12^2 = 16^2 + 10^2 + 3^2 = 14^2 + 12^2 + 5^2.$$

(b) Prove that the sum of the squares of three consecutive positive integers, each greater than 3, can be written as the sum of squares of three non-consecutive positive integers.

Solution. Since the beginning integer is of form $3n$, $3n+1$ or $3n+2$, one of (1), (2) or the formula derived in (a)(iii) can be used to write the sum of the squares of three consecutive positive integers as a sum of squares of three non-consecutive integers. Since each number is greater than 3, $n > 2$ in (1) and (2) or $n > 1$ for the formula of (a)(iii), guaranteeing non-zero righthand side squares.

(c) Find all the ways to write $7^2+8^2+9^2$ as the sum of three squares of positive integers.

Solution. We see that the right hand side cannot have a square larger than 13^2. Searching systematically with decreasing squares we obtain

$$\begin{aligned}
7^2 + 8^2 + 9^2 &= 13^2 + 4^2 + 3^2\\
&= 12^2 + 7^2 + 1^2\\
&= 12^2 + 5^2 + 5^2\\
&= 11^2 + 8^3 + 3^2.
\end{aligned}$$

10. Let $ABCD$ be a parallelogram with P a point inside it such that $\angle APB+\angle CPD = 180°$. Prove that $\angle PBC = \angle CDP$.

Investigation.
If $ABCD$ is a rhombus, what is the locus of P?