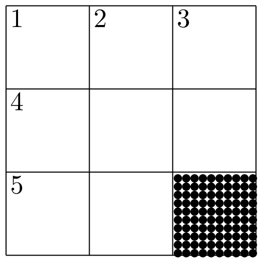
The University of Western Australia
DEPARTMENT OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2004 Australian Intermediate Mathematics Olympiad Problems

1. The following cross-number puzzle has one digit in each square:



   1 across and **1 down** are squares of different primes.
   5 across and **2 down** are different squares.
   3 down is a prime.

   What is **4 across**?

2. A polygonal prism has 2004 edges, How many faces does it have?

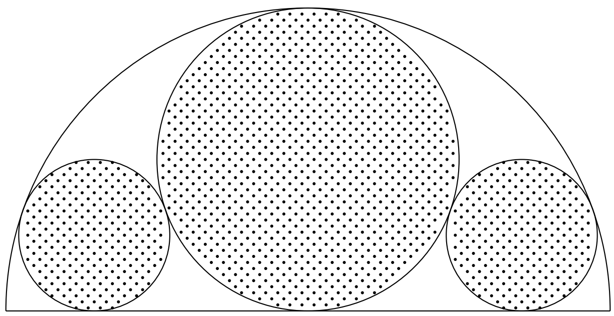
3. Pamela grows pumpkins. She digs up nine small pumpkins and calculates their mean mass. Next, Pamela digs up her prize specimen, a very large pumpkin, and adds it to the other nine. She discovers that the mean mass of her pumpkin harvest has doubled. Let the very large pumpkin have mass $P\%$ of the total mass of all ten pumpkins.

   Find P.

4. David is swimming upstream from a dock at a constant rate. After he has swum 400 metres, he meets Julie who is floating downstream on an air-mattress. He continues to swim upstream for a further 10 minutes before turning round and swimming back to the dock, which he reaches at the same time as Julie. Assume David's swimming speed is constant relative to the water and that the current has constant speed of x metres per minute.

   Find x.

5. The diagram below shows three circles externally tangent to each other and to a semicircle. The shaded area is 120.



   What is the area of the unshaded parts of the semicircle?

6. A new binary operation $*$, defined for all $x, y \in \mathbb{N}$, satisfies:

   (i) $x * x = x + 2$,

   (ii) $x * y = y * x$,

   (iii) $\dfrac{x * (x+y)}{x * y} = \dfrac{x+y}{y}$.

   Find $11 * 8$.

7. Integers $a, b, c \in \{1, 2, 3, \ldots, 10\}$ are respectively written on three cards.

   The three cards are shuffled, dealt one each to three players, who record the number on the card they receive, and then the cards are collected. The process is repeated a few times.

   Then the players compute the totals of the numbers they had received. The totals were 13, 15 and 23.

   Find the value of abc.

8. AB and CD are chords of a circle of radius 4, on opposite sides of the centre of that circle. CB and DA produced meet at P, outside the circle. The length of arc CD is four times that of arc AB. The length of arc CD is $\frac{8\pi}{5}$.

   Find the number of degrees in $\angle APB$.

9. The side AB of $\triangle ABC$ is a diameter of a circle of radius R and C lies on this circle. The angle bisector of $\angle BAC$ meets BC at E and then the circle at D. When produced, AC meets the circumcircle of $\triangle CED$ at F.

   If $BC = a$, express CF in terms of R and a.

10. When the numerator and denominator of the fraction $\frac{1}{2}$ are each increased by 2, the fraction $\frac{3}{4}$ is produced.

    (a) Find three more fractions, which are in lowest terms, each with the property that when their denominators are increased by 2, the fraction $\frac{3}{4}$ is produced.

    Let $\frac{x}{y}$ be a fraction, in lowest terms, with the property that when its numerator and denominator are each increased by 2, the fraction $\frac{3}{4}$ is produced.

    (b) Find a formula for $\frac{x}{y}$ in the form $\frac{x}{y} = \dfrac{an+b}{cn+d}$, $n = 0, 1, \ldots$, which produces all fractions with the property, where $a, b, c, d \in \mathbb{Z}$.

    Note. You must prove that your formula produces all fractions with the property.

    Investigation. Starting with $\frac{1}{2}$, increasing its numerator and denominator by $k \in \mathbb{N}$ will produce another fraction f, say. There will be many fractions, each with the property that when their numerators and denominators are increased by k, the fraction f is produced. Investigate this situation for a value or values of k other than 2.