The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2005 Australian Intermediate Mathematics Olympiad Problems

1. Consider a cube of edge 9 cm. In the centre of three different and not opposite faces, a square hole is made which goes through to the opposite face. Each side of each hole has length 3 cm.

   What is the surface area, in cm^2, of the remaining solid?

2. Water taxis which take 6, 10 or 15 passengers are used to transport passengers from an airport to a hotel. Six water taxis of each size were available when a party of 120 arrived.

   In how many different ways is it possible to use the taxis, so that no taxi used has an empty seat?

3. Andrew, Brian, Charles and David all weigh different amounts. Andrew is 8 kg heavier than Charles and David is 4 kg heavier than Brian. The sum of the masses of the heaviest and lightest men is 2 kg less than the sum of the masses of the other two. The total mass of all four men is 402 kg.

   How many kg is the mass of Andrew?

4. Let $A_1A_2 \ldots A_{15}$ be a regular pentadecagon (a 15-sided polygon). Line ℓ contains the interval A_1A_2, and each of the sides not adjacent to ℓ are extended to intersect ℓ. The acute angle of the 12 intersections is calculated.

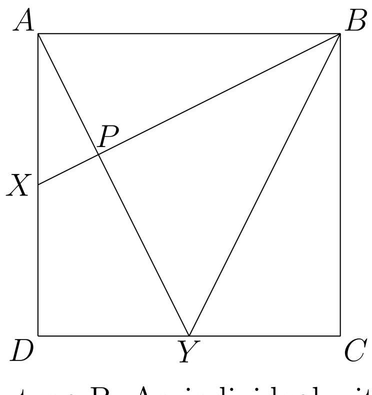
   Find the sum in degrees of the 12 acute angles formed.

5. A positive integer n is such that its only digits are 3s, and such that $383 \mid n$.

   When $\frac{n}{383}$ is divided by 1000, what is the remainder?

6. A square $ABCD$ has side length 10. X and Y are midpoints of sides AD and CD, respectively.

   What is the total area of triangles XPA and YBP, where P is the point of intersection of AY and XB?



7. A disease can be of two types: type A, or the less common type B. An individual with the disease can have only one of these two types.

   In a certain population, 1 in 320 individuals suffer from the disease. Suppose that 1 in x of the population suffer from type A and 1 in y of the population suffer from type B, where $x, y \in \mathbb{N}$.

   What is the smallest possible value of y?

8. It was Arthur's and Martha's turn to hold the pre-Christmas dinner. The two hosts drew up a list of friends to invite.

   Arthur contributed 13 male names to the list and Martha added 15 female names. Everyone on the list was asked to come with a partner and they all did so. Each person, including each partner, was also asked to bring a small gift for each dinner guest, except for themselves and their partner.

   After everyone arrived, Martha suggested that they exchange gifts and use this as a way of meeting and greeting. However, after a short time, the first dinner course was ready and Martha had to interrupt the exchange of gifts. Before going into the dining room, she asked, out of curiosity, how many gifts each individual had exchanged up to that stage. Everyone, including Arthur (but not Martha herself) answered, and amazingly every number was different!

   How many gifts had Arthur exchanged?

9. A rectangle $ABCD$ has $AB = 10$ and $BC = 8$. Let L be the point on AB such that $AL = 1$, and let M, N and O be points on BC, CD and AD, respectively, such that $LMNO$ is a rectangle.

   (a) Show that there are two possible positions for M.

   (b) Let R_1 be the area of the smaller inner rectangle and R_2 the area of the larger inner rectangle.

      (i) Show that $R_1 + R_2 = (ABCD)$, where $(ABCD)$ represents the area of $ABCD$.

      (ii) Find the ratio $\frac{R_1}{R_2}$ in simplest surd form with a rational denominator.

   Now let $AL = x$.

   (c) Show that there are values of x for which there is only one position for M.
   What is this position and what is the area of $LMNO$ for these values of x?

   (d) Find all values for x for which there is

      (i) no position possible for M, i.e. no rectangle $LMNO$ can be formed; and

      (ii) more than one possible position for M.

10. (a) A circle is inscribed in quadrilateral $ABCD$. The sides BC and DA have different lengths, and sides AB and CD are parallel with lengths a and b, respectively.

      (i) Prove that the quadrilateral is cyclic.

      (ii) Find the length of the diameter of the inscribed circle in terms of a and b.

      (iii) Prove that the square of the area of the quadrilateral is equal to the product of the lengths of its sides.

   (b) Prove that if a circle can be inscribed in a quadrilateral, then the sums of the two pairs of opposite sides are equal.

   Investigation. State the converse of the theorem in (b).

   Is the converse true? Prove or disprove the converse.