The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2006 Australian Intermediate Mathematics Olympiad Problems with Some Solutions

1. Find $a+b+c+d$ given

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}}=\frac{11}{42},$$

   where a,b,c,d are positive integers.

2. Find the number of solutions of the equation

$$x^2y^3=6^{12},$$

   where x and y are positive integers.

3. The wealthy and education-minded local community of Misery Creek decided to add five extra classrooms to its secondary college, which enabled there to be five more classes and led to a reduction of the school's average class size by 6.

   Two months later, due to an unexpected income from mining royalties, the community had another five classrooms added to the college, enabling five more classes to be formed and a consequent reduction in the average class size by 4. During each of these construction periods, the number of students in the school did not change.

   What was the number of students at Misery Creek Secondary College?

4. The letters E, H, I, N, O, R, S, T, W and X represent separately the digits $0,1,\ldots,9$, but not necessarily in that order.

$$\text{ONE}\times\text{TWO}=\text{THREE}.$$

   The square root of TWO and the square root of SIX are twin primes, that is, their difference is 2.

   Find the number TEN.

5. Three pipes lead into a dam. The pipes are called Upper, Lower and Middle by the owner of the property on which the dam was situated. He found that he could fill the dam in a number of ways:

   The lower and upper pipes flow for 3 days.

   The middle and upper pipes flow for 4 days.

   The lower and middle pipes flow for 6 days.

   Assume each pipe has water flowing at a constant rate which is not affected by the flow in other pipes. How many hours does it take to fill the dam if three pipes are flowing?

Solution.

This is essentially the same as the following 'cow, sheep, goat' problem:

> A cow and a sheep eat out a paddock in 45 days.
> A cow and a goat eat out a paddock in 60 days.
> A sheep and a goat eat out a paddock in 90 days.
>
> How long does it take for the three animals together to eat out the paddock?
> [Assume that there is a fixed amount of grass in the paddock.]
>
> **Solution.** Let c, s, g, respectively, be the number of days that the cow, sheep and goat would take to eat out the paddock individually. Thus, e.g. in one day the fraction of a paddock eaten by a cow is $1/c$, so that we have the following equations from the given information:
>
> $$\frac{1}{c} + \frac{1}{s} = \frac{1}{45}$$
> $$\frac{1}{c} + \frac{1}{g} = \frac{1}{60}$$
> $$\frac{1}{s} + \frac{1}{g} = \frac{1}{90}$$
>
> Adding these three equations we get
>
> $$\begin{aligned} 2\Big(\frac{1}{c} + \frac{1}{s} + \frac{1}{g}\Big) &= \frac{1}{45} + \frac{1}{60} + \frac{1}{90} \\ &= \frac{4+3+2}{180} = \frac{1}{20} \\ \frac{1}{c} + \frac{1}{s} + \frac{1}{g} &= \frac{1}{40} \end{aligned}$$
>
> i.e. in one day the three animals eat out $1/40$ of a paddock, and hence will take 40 days to eat out the paddock.

For our problem, let ℓ, m, u, respectively, be the number of days it will take for flow through the Lower, Middle and Upper pipes would take to fill the dam, individually. Then from the given information we have:

$$\frac{1}{\ell} + \frac{1}{u} = \frac{1}{3}$$
$$\frac{1}{u} + \frac{1}{m} = \frac{1}{4}$$
$$\frac{1}{\ell} + \frac{1}{m} = \frac{1}{6}$$

Adding these three equations we get

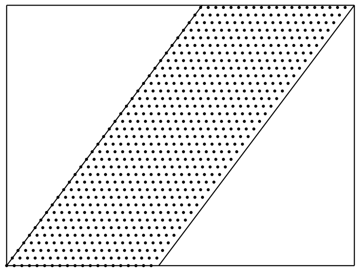
$$\begin{aligned} 2\Big(\frac{1}{\ell} + \frac{1}{m} + \frac{1}{u}\Big) &= \frac{1}{3} + \frac{1}{4} + \frac{1}{6} \\ &= \frac{4+3+2}{12} = \frac{3}{4} \\ \frac{1}{\ell} + \frac{1}{m} + \frac{1}{u} &= \frac{3}{8} \end{aligned}$$

i.e. in one day flow through the three pipes will fill $3/8$ of the dam and hence it will take

$$\frac{8}{3}\,\text{day} = \frac{8}{3} \cdot 24\,\text{h} = 64\,\text{h}$$

to fill the dam.

6. The Ruritanian flag, rectangular in shape, consists of a red stripe on a white background as in the diagram. The slant edges of the stripe extend from two of the opposite corners to the opposite sides as shown, and each of these slant edges is perpendicular to the diagonal of the rectangle that joins its remaining corners. One of the flag sizes is 40 cm by 30 cm. Find the area of the red stripe of this flag in cm^2.



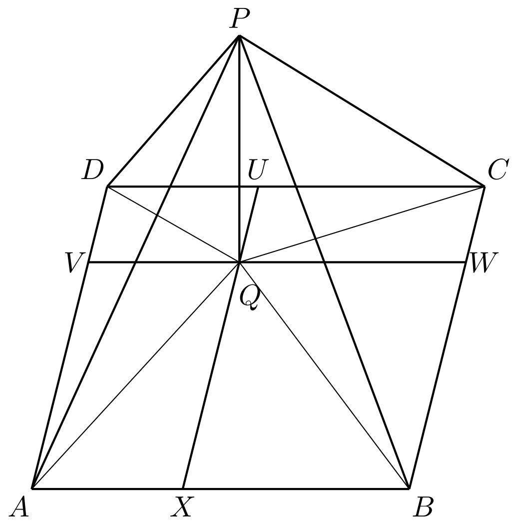
7. N is a 4-digit perfect square with all its digits less than 7. When each digit is increased by 3, another perfect square is obtained. Let $N = n^2$. Find n.

Solution. Let $m^2 = M$ be the other square. Then $m^2 = M = N + 3333$, and $N = n^2$.
Now $3.11.101 = 3333 = M - N = m^2 - n^2 = (m-n)(m+n)$.
Also $1000 \le n^2 < m^2 < 10\,000$. So $32 \le n < m \le 99$.
Hence $65 = 33 + 32 \le m + n \le 99 + 98 = 197$. $\therefore m + n = 101, m - n = 33$.
So $m = \frac{1}{2}(101 + 33) = 67$, $n = 67 - 33 = 34$.

8. $ABCD$ is the rectangular base of a pyramid whose vertex is P. If $PA = 670$, $PB = 938$ and $PC = 737$, find PD.

Solution.
Draw Q as the foot of the perpendicular from p to the base of the pyramid. Let U, V, W and X be points on CD, DA, AB and BC, respectively, such that $UX \parallel DA$ and $VW \parallel AB$, and let Q be the intersection point of UX and VW.
Then, by Pythagoras' Theorem,

$$\begin{aligned}
PA^2 &= PQ^2 + QA^2 = PQ^2 + QX^2 + QV^2 \\
PB^2 &= PQ^2 + QB^2 = PQ^2 + QX^2 + QW^2 \\
PC^2 &= PQ^2 + QC^2 = PQ^2 + QU^2 + QW^2 \\
PD^2 &= PQ^2 + QD^2 = PQ^2 + QU^2 + QV^2
\end{aligned}$$



Now observe that

$$\begin{aligned}
PD^2 &= PQ^2 + QU^2 + QV^2 \\
&= PA^2 + PC^2 - PB^2 \\
&= 670^2 + 737^2 - 938^2 \\
&= 670^2 + (737 + 938)(737 - 938) \\
&= 670^2 - 1675 \cdot 201 \\
&= 670^2 - 5^2 \cdot 67^2 \cdot 3 \\
&= 67^2 \cdot 5^2(4 - 3) \\
&= 335^2
\end{aligned}$$

$\therefore PD = 335.$

9. Let $p \leq q \leq r$ be prime numbers such that $pqr(p+q+r)$ is a perfect square. What is the largest value of $p+q+r$?

Solution. Let $S = pqr(p+q+r)$. Consider 4 cases:

Case 1. $p = q = r$. Then $S = 3p^4$ which is not a perfect square.#

Case 2. $p = q < r$. Then $S = p^2r(2p+r)$. For S to be a perfect square, $r(2p+r)$ must be a perfect square, and so $r \mid 2p$. But $r > p \implies r \nmid p$. Hence $r \mid 2$, so that $r = 2$. But then $p = q < 2$, so that p, q are not prime.#

Case 3. $p < q = r$. Then $S = pq^2(p+2q)$. For S to be a perfect square, $p(p+2q)$ must be a perfect square, and so $p \mid 2q$. Similarly to the previous case, we have $p = 2$, so that $S = 4q^2(q+1)$ and hence $q+1 = s^2$ (since it must be a perfect square). Thus $q = s^2 - 1 = (s-1)(s+1)$. Since q is prime, $s - 1 = 1$, i.e. $s = 2, q = 3$.
Thus we have $p = 2, q = r = 3$ giving $S = 144$ (a perfect square) and $p+q+r = 8$.

Case 4. $p < q < r$. For S to be a perfect square, $r \mid p+q+r$, and hence $r \mid p+q < 2r$ and so so $p+q = r$.
Then $S = 2pqr^2$, in which case $p = 2$, since $2 \mid S$ implies $2^2 \mid S$, and this leaves us with q being a square, which is not the case.#.

So, in fact, there is exactly one solution (and so it is the least!), namely $p+q+r = 8$.

10. In triangle ABC, D is a point on BC. The incircles of triangles ADB and ADC are both tangent to AD at the same point. Show that the incircle of ABC touches BC at D.

Investigation:

Describe ABC and the point D if, in addition to the conditions in the problem, the radii of ABD and ACD are equal. Prove your statements.