The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

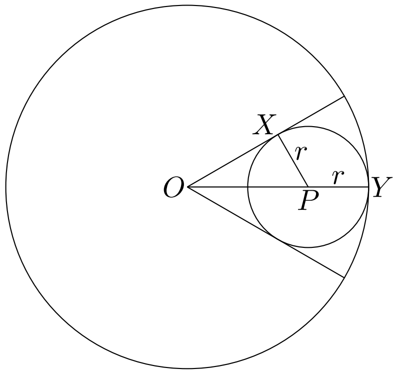
2008 Australian Intermediate Mathematics Olympiad Problems with Solutions

1. Consider a circular sector of radius 360 which is one-sixth of a circle. A circle is drawn inside this sector so that its is tangent to the two radii and to the circular arc. Calculate the radius of this smaller circle.

   Solution. Let O be the centre of the large circle, P and r be the centre and radius, respectively, of the small circle (call it K). Let X be one of the points where the sector meets the small circle tangentially, and Y the point where the small circle meets the large circle. Since the sector is one-sixth of the large circle, the angle subtended at O is $\frac{1}{6}\cdot 360° = 60°$. Hence,

   $$\angle XOP = \tfrac{1}{2}\cdot 60° = 30°$$

   Also, $\angle PXO = 90°$, since OX is tangent to K at X



   Thus, $\triangle OXP$ is a half-equilateral triangle, and hence $PX : XO : OP = 1 : \sqrt{3} : 2$. So

   $$\begin{aligned}\frac{r}{360-r} = \frac{PX}{OP} &= \frac{1}{2}\\ 2r &= 360 - r\\ 3r &= 360\\ r &= 120\end{aligned}$$

   Therefore, the radius of the small circle is 120.

2. Find the 3-digit number at the right-hand end of $1! + 2! + 3! + \cdots + 2008!$.

   Solution. To find the 3-digit number at the right-hand end of a natural number N is equivalent to reducing N *modulo* 1000. Observe that $2^3 = 2\cdot 4 \mid 4!$ and that $5^3 \mid 5\cdot 10\cdot 15 \mid 15!$.

   $$\therefore 1000 = 2^3\cdot 5^3 \mid 15! \mid 16! \mid \cdots \mid 2008!$$
   $$\text{Also, } 13! + 14! = (1+14)\cdot 14!$$
   $$\therefore 2^3\cdot 5^3 \mid (13! + 14!)$$
   $$\begin{aligned}\therefore 1! + 2! + \cdots + 2008! &\equiv 1! + 2! + \cdots + 12! \pmod{1000}\\ &\equiv 1 + 2 + 6 + 24 + 120 + 720 + 320 + 880 + 800 + 800 + 600 \pmod{1000}\\ &\equiv 313 \pmod{1000}\end{aligned}$$

   Hence, the 3-digit number at the right-hand end of $1! + 2! + 3! + \cdots + 2008!$ is 313.

3. If $\dfrac{2}{35} = \dfrac{1}{x} + \dfrac{1}{y}$ and x, y are different positive integers, find the minimum value of $x + y$.

Solution.

$$\frac{2}{35} = \frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy}$$
$$\implies \begin{cases} x + y = 2k \implies y = 2k - x \\ xy = 35k \implies x(2k - x) = 35k \end{cases}$$
$$\therefore\ x^2 - 2kx + 35k = 0 \tag{1}$$

By symmetry, y also satisfies (1).

$$\therefore\ x, y = \frac{2k \pm \sqrt{2k^2 - 4 \cdot 35k}}{2} = k \pm \sqrt{k^2 - 35k} \tag{2}$$

Since $x, y \in \mathbb{Z}$, $k^2 - 35k = k(k-35)$ is a square.
Since x, y are distinct, $k(k-35) > 0 \implies k \geq 36$.
But for $k = 36$, $k(k-35) = 36$ is a square.
Now, x, y are the roots of (2). In general, for a monic polynomial with zeros α, β

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$$

i.e. the coefficient of x is the sum of the zeros. Applying this to (2) we have

$$x + y = 2k$$
$$\therefore \min(x + y) = \min(2k) = 2 \cdot 36 = 72$$

So the minimum value of $x + y$ is 72.

4. Find the largest prime factor of $7^{14} - 56 + 7^{13}$.

Solution. Let N be the number. Then

$$\begin{aligned}
N &= 7^{14} - 56 + 7^{13} \\
&= 7(7^{13}7^{12} - 8) \\
&= 7\big(7^{12}(7+1) - 8\big) \\
&= 7 \cdot 2^3 \cdot (7^{12} - 1) \\
&= 7 \cdot 2^3 \cdot (7^6 - 1)(7^6 + 1) \\
&= 7 \cdot 2^3 \cdot (7^3 - 1)(7^3 + 1)(7^2 + 1)(7^4 - 7^2 + 1) \\
&= 7 \cdot 2^3 \cdot (7 - 1)(7^2 + 7 + 1)(7 + 1)(7^2 - 7 + 1) \cdot 50 \cdot \big(7^2(7^2 - 1) + 1\big) \\
&= 7 \cdot 2^3 \cdot 6 \cdot 57 \cdot 2^3 \cdot 43 \cdot 2 \cdot 5^2 \cdot (49 \cdot 48 + 1), \quad 49 \cdot 48 + 1 = (50 - 1)(50 - 2) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 2500 - 150 + 2 \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 2352 \\
&= 2^8 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 19 \cdot 43 \cdot 2353 \\
&= 2^8 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 19 \cdot 43 \cdot 13 \cdot 181
\end{aligned}$$

where in factorising 2353 we tried $2, 3, 5, 7, 11$ before finding that $13 \mid 2353$. Since $2, 3, 5, 7, 11$ each do not divide 2353, they do not divide 181, and since also $13 \nmid 181$, 181 is prime.

$\therefore$ 181 is the largest prime factor of N.

5. Each interior angle in a 16-sided convex polygon is an integer number of degrees. When arranged in ascending order of magnitude, these angles form an arithmetic progression. How many degrees are there in the largest interior angle in the polygon?

Solution. The sum of interior angles for an n-gon is $180(n-2)^\circ$. Let the smallest angle be a, the largest angle be ℓ and the common difference for the arithmetic progression (of angles) be d. Then

$$\begin{aligned}\text{Sum of angles} &= 180(16-2)\\ &= 14\cdot 180\\ &= \frac{16}{2}(a+\ell),\ \text{ where } \ell = a + 15d\\ \therefore 315 = 7\cdot 45 &= a+\ell\end{aligned}$$

Now each of the angles is an integer. So $d \in \mathbb{N}$. Since $\ell - a = 15d$, $15 \mid \ell - a$. The 16-gon is convex.

$$\begin{aligned}\therefore \ell = 315 - a &< 180\\ \therefore a &> 315 - 180 = 135\\ \text{i.e. } \ell &= 180 - k \text{ and}\\ a &= 135 + k,\ \text{ for some integer } k \geq 1\\ \therefore 15 \mid \ell - a &= 45 - 2k\\ \therefore 45 - 2k &= \begin{cases} 30 \implies k = \frac{15}{2} \implies a \notin \mathbb{Z} \text{ (contradiction)}\\ 15 \implies k = 15 \implies a = 150, \ell = 165\\ \ \ 0 \implies k = \frac{45}{2} \implies a \notin \mathbb{Z} \text{ (contradiction)}\end{cases}\end{aligned}$$

Hence the largest angle $\ell = 165^\circ$ (and the smallest angle is $a = 150^\circ$).

6. Impatient Imran always walks down the moving escalator outside his office. It moves at a constant but annoyingly slow speed. Once he got from top to bottom in 16 seconds taking 28 steps. Another time he got from top to bottom in 24 seconds taking 21 steps. How many steps high is the escalator?

Solution. Let

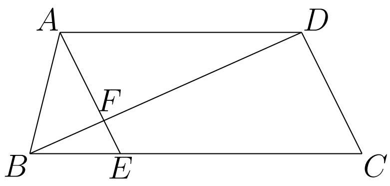
$$\begin{aligned} s &= \text{number of steps of escalator}\\ v &= \text{speed of escalator in steps/s.}\end{aligned}$$

Then

$$\begin{aligned}\frac{s-28}{v} &= 16 \implies s - 28 = 16v\\ \frac{s-21}{v} &= 24 \implies s - 21 = 24v\\ \therefore \tfrac{3}{2}(s-28) &= s - 21\\ \therefore 3(s-28) &= 2(s-21), \qquad \text{eliminating } v\\ s &= 3\cdot 28 - 2\cdot 21\\ &= 42 \text{ steps}\end{aligned}$$

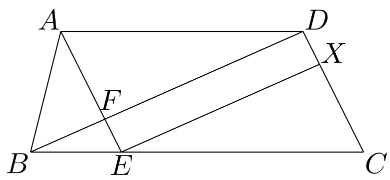
Hence the escalator is 42 steps high.

7. In the trapezium $ABCD$, $AD \parallel BC$, $BD \perp DC$, the point F is chosen on diagonal BD so that $AF \perp BD$, and AF is extended to meet BC at the point E. If $AB = 41$, $BF = 9$ and the area of quadrilateral $FECD$ is 960, what is the length of AD?



Solution. Draw $EX \parallel FD$ so that $DX = FE$. Let $x = FD$ and let $y = FE$. Also (XYZ) represents the area of figure XYZ.

$$\begin{aligned} AF^2 &= AB^2 - BF^2 \\ &= 41^2 - 9^2 \\ &= 40^2, && \text{(triad: } 9:40:41) \\ \therefore AF &= 40 \\ AE &\parallel DC, && \text{since } AF \perp BD \perp DC \end{aligned}$$



Therefore, $AECD$ is a parallelogram.

$$\begin{aligned} \therefore AE &= DC \text{ and } FEXD \text{ is a rectangle} \\ \therefore XC &= DC - DX \\ &= AE - FE \\ &= AF = 40 \\ \angle FBE &= \angle DBC, && \text{(same angle)} \\ \angle BFE &= \angle BDC = 90^\circ \\ \therefore \triangle BFE &\sim \triangle BDC, && \text{(by AA Rule)} \\ \therefore \frac{9+x}{9} &= \frac{40+y}{y}, && \text{where } x = FD,\ y = FE = DX \\ 1 + \frac{x}{9} &= 1 + \frac{40}{y} \\ xy &= 9 \cdot 40 = 360 = (FDXE) \\ \therefore (EXC) = \frac{40x}{2} &= 960 - 360 = (FECD) - (FDXE) \\ \therefore x &= \frac{600}{20} = 30 \\ \therefore AD^2 &= \sqrt{30^2 + 40^2} = 50 \\ \therefore AD &= 50 \end{aligned}$$

8. Curious Kate calculated the sum of all positive integers each of which equals 101 times the sum of its digits. Find the remainder when her sum is divided by 1000.

Solution. Let $(a_n a_{n-1} \dots a_0)$ be the decimal representation of one of Kate's numbers. Then

$$\begin{aligned} \sum_{k=0}^{n} a_k \cdot 10^k &= 101 \sum_{k=0}^{n} a_k \\ \implies 0 &= \sum_{k=0}^{n} a_k \cdot (10^k - 101) \qquad (3) \\ &= -100a_0 - 91a_1 - a_2 + 899a_3 + 9899a_4 + \cdots \end{aligned}$$

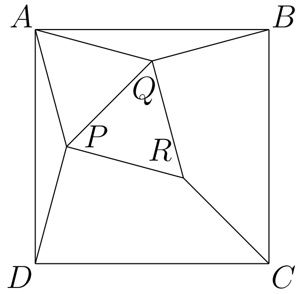
If $n \le 2$ then RHS of (3) < 0 (contradiction).
If $n \ge 4$ then RHS of (3) $\ge 9899 - 100 \cdot 9 - 1 \cdot 9$
$= 9899 - 1728$
> 0 (contradiction).

$\therefore n = 3$.
If $a_3 \ge 2$ then RHS of (3) $\ge 899 \cdot 2 - 1728 > 0$ (contradiction).
$\therefore a_3 = 1$.

$$\begin{aligned} \therefore 100a_0 + 91a_1 + a_2 &= 899 \\ \therefore a_1 + a_2 &\equiv 9 \pmod{10} \\ \therefore a_1 + a_2 &= 9, && \text{since } 0 \le a_1, a_2 \le 9 \\ \therefore 100a_0 + 90a_1 + (a_1 + a_2) &= 890 + 9 \\ \therefore 10a_0 + 9a_1 &= 89 \\ \therefore 9a_1 &\equiv 9 \pmod{10} \\ \therefore a_1 &\equiv 1 \pmod{10}, && \text{since } (9, 10) = 1 \\ \therefore a_1 &= 1 \text{ and } a_2 = 8 \\ \therefore a_0 &= 8 \end{aligned}$$

i.e. there is only one choice: $a_0 = 8$, $a_1 = 1$, $a_2 = 8$, $a_3 = 1$.
So Kate has just the one number: 1818. So the sum of such numbers is also 1818 and the last 3 digits of the sum is 818.

9. $ABCD$ is a square. P, Q. R are points such that $\triangle APQ$ and $\triangle PQR$ are equilateral, and $AQ = QB$ and $AP = PD$. Prove that $RC = PD$.



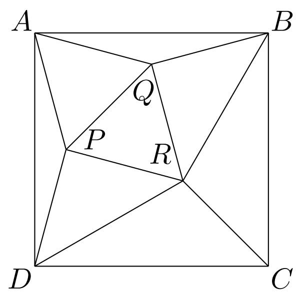
Solution. Since $\triangle APQ$ and $\triangle PQR$ are equilateral, and $AQ = QB$ and $AP = PD$,

$$QR = PR = QP = AP = AQ = QB = PD = x, \text{ say.}$$

Since $ABCD$ is a square,

$$AB = BC = CD = DA = y, \text{ say.}$$

Thus we have



$$\begin{aligned} AQ &= AP = x \\ QB &= PD = x \\ AB &= AD = y \\ \therefore \triangle AQB &\cong \triangle APD, && \text{by the SSS Rule} \\ &\text{and } \triangle AQB, \triangle APD \text{ are isosceles} \\ \therefore \angle QBA = \angle QAB = \angle PAD &= \tfrac{1}{2}(\angle DAB - \angle PAQ) \\ &= \tfrac{1}{2}(90^\circ - 60^\circ) = 15^\circ \\ \therefore \angle AQB &= 180^\circ - 2 \cdot 15^\circ = 150^\circ \\ \angle BQR &= 360^\circ - \angle AQB - \angle AQP - \angle PQR \\ &= 360^\circ - 150^\circ - 60^\circ - 60^\circ = 90^\circ \\ \triangle BQR &\text{ is isosceles, since } QR = QB = x \\ \therefore \angle QBR = \angle QRB &= \tfrac{1}{2}(180^\circ - 90^\circ) = 45^\circ \\ \text{and } BR &= x\sqrt{2}, && \text{(isosceles right triangle forms triad } 1:1:\sqrt{2}) \\ \therefore \angle RBC &= \angle ABC - \angle QBA - \angle QBR \\ &= 90^\circ - 15^\circ - 45^\circ = 30^\circ \end{aligned}$$

Similarly,

$$
\begin{aligned}
DR &= x\sqrt{2} \text{ and } \angle RDC = 30^\circ \\
\therefore DR &= BR = x\sqrt{2} \\
DC &= BC = y \\
RC & \text{ is common} \\
\therefore \triangle DRC &\cong \triangle BRC, && \text{by the SSS Rule} \\
\therefore \angle RCD &= \angle RCB = \tfrac{1}{2} \cdot 90^\circ = 45^\circ \\
\therefore \frac{RC}{\sin 30^\circ} &= \frac{x\sqrt{2}}{\sin 45^\circ}, && \text{by the Sine Rule applied to } \triangle BRC \\
\therefore RC &= \tfrac{1}{2} \cdot x\sqrt{2} \cdot \frac{1}{\sqrt{2}} = x \\
\therefore RC &= PD
\end{aligned}
$$

10. Real numbers a, b, c, d, e are linked by the two equations:

$$
\begin{aligned}
e &= 40 - a - b - c - d \\
e^2 &= 400 - a^2 - b^2 - c^2 - d^2.
\end{aligned}
$$

Determine the largest value for e.

Investigation

Find all integer solutions, if any, of the following pair of equations.

$$
\begin{aligned}
e &= 30 - a - b - c - d \\
e^2 &= 200 - a^2 - b^2 - c^2 - d^2.
\end{aligned}
$$

Solution. Rearranging the given equations we obtain the following symmetric forms

$$
\begin{aligned}
a + b + c + d + e &= 40 && (4) \\
a^2 + b^2 + c^2 + d^2 + e^2 &= 400 && (5)
\end{aligned}
$$

Now we perform a few manipulations aimed at reducing the problem.

$$
\begin{aligned}
(a-8) + (b-8) + (c-8) + (d-8) + (e-8) &= 0, && \text{rearranging (4)} \\
a^2 - 16a + b^2 - 16b + c^2 - 16c + d^2 - 16d + e^2 - 16e &= 400 - 16 \cdot 40, && (5) - 16 \cdot (4) \\
(a-8)^2 + (b-8)^2 + (c-8)^2 + (d-8)^2 + (e-8)^2 &= 400 - 16 \cdot 40 + 5 \cdot 64, && \text{completing the square 5 times} \\
&= 80
\end{aligned}
$$

Now letting $A = a - 8$, $B = b - 8$, $C = c - 8$, $D = d - 8$ and $E = e - 8$, the problem is reduced to maximising E where

$$
\begin{aligned}
A + B + C + D + E &= 0 && (6) \\
A^2 + B^2 + C^2 + D^2 + E^2 &= 80 && (7)
\end{aligned}
$$

Background
AM-GM Theorem. For a sequence of non-negative numbers $x_1, x_2, \ldots, x_n$ its arithmetic mean (AM) is $\geq$ its geometric mean (GM), i.e.

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Thus by AM-GM, for $n = 2$, we have

$$\frac{A^2 + B^2}{2} \geq AB, \ \frac{A^2 + C^2}{2} \geq AC, \ \ldots, \ \frac{C^2 + D^2}{2} \geq CD,$$

with these all becoming *equalities* if $A = B = C = D$. We will use this in step (9) below. Isolating E in (6) we have

$$E = -(A + B + C + D) \tag{8}$$

$$\begin{aligned} E^2 &= (A + B + C + D)^2 \\ &= A^2 + B^2 + C^2 + D^2 + 2AB + 2AC + \cdots + 2CD \end{aligned}$$

$$\leq A^2 + B^2 + C^2 + D^2 + (A^2 + B^2) + (A^2 + C^2) + \cdots + (C^2 + D^2) \tag{9}$$

$= 4(A^2 + B^2 + C^2 + D^2)$, since from $2AB$, $2AC$ and $2AD$ we obtain $3A^2$ and by symmetry there are as many A^2s as B^2s, C^2s and D^2s

$= 4(80 - E^2)$, using (7)

$$\begin{aligned} \therefore 5E^2 &\leq 4 \cdot 80 \\ E^2 &\leq 4 \cdot 16 \\ E &\leq 8, \qquad \text{with equality if } A = B = C = D \ (= -E/4 \text{ by } (8)\,) \end{aligned}$$

Therefore,

the maximum value of E is 8 when $A = B = C = D = -2$

i.e. the maximum value of e is 16 when $a = b = c = d = 6$

(since $a - 8 = A, \ldots, e - 8 = E \implies a = A + 8, \ldots, e = E + 8$).

We start the *investigation* problem in a similar way. Rearranging the given equations we obtain the following symmetric forms

$$a + b + c + d + e = 30 \tag{10}$$

$$a^2 + b^2 + c^2 + d^2 + e^2 = 200 \tag{11}$$

Now we perform a few manipulations aimed at reducing the problem.

$(a-6) + (b-6) + (c-6) + (d-6) + (e-6) = 0,$	rearranging (10)
$a^2 - 12a + b^2 - 12b + c^2 - 12c + d^2 - 12d + e^2 - 12e = 300 - 12 \cdot 30,$	$(11) - 12 \cdot (10)$
$(a-6)^2 + (b-6)^2 + (c-6)^2 + (d-6)^2 + (e-6)^2 = 20,$	completing the square 5 times

Now letting $A = a - 6$, $B = b - 6$, $C = c - 6$, $D = d - 6$ and $E = e - 6$, the problem is reduced to finding integer solutions of

$$A + B + C + D + E = 0 \tag{12}$$

$$A^2 + B^2 + C^2 + D^2 + E^2 = 20 \tag{13}$$

Since the equations are symmetric in A, B, C, D, E we may assume without loss of generality that

$$|A| \geq |B| \geq |C| \geq |D| \geq |E|$$

(and then take all permutations of the solutions we find to obtain all solutions, later). By (13), $A^2 \leq 20$ and hence $|A| \leq 4$. If $|A| = 4$, then $B^2 \leq 4 = 20 - 16$, whence $|B| = 2$ and $C = D = E = 0$, or $|B| = |C| = |D| = |E| = 1$. We then need to check if (12) can be satisfied. We continue in this systematic way to find all solutions, and tabulate the investigation below. A # in a column indicates the relevant condition cannot be satisfied.

$\|A\|$	$\|B\|$	$\|C\|$	$\|D\|$	$\|E\|$	(A, B, C, D, E) s.t. $A + B + C + D + E = 0$
4	2	0	0	0	#
4	1	1	1	1	$A = 4, \quad B = C = D = E = -1$ or $A = -4, \quad B = C = D = E = 1$
3	3	1	1	0	$A = -B, \quad C = -D$
3	2	2	1	#	#
2	2	2	2	2	#

At line 3 we impose the extra condition $A > 0$ and $C > 0$ since a permutation will get the other possibilities. Thus, up to permutation, we have 3 solutions

$$\begin{aligned} A = 4,\ B = C = D = E = -1 &\implies a = 10,\ b = c = d = e = 5 \\ A = -4,\ B = C = D = E = 1 &\implies a = 2,\ b = c = d = e = 7 \\ A = 3,\ B = -3,\ C = 1,\ D = -1,\ E = 0 &\implies a = 9,\ b = 3,\ c = 7,\ d = 5,\ e = 6 \end{aligned}$$

i.e. the complete set of solutions (a, b, c, d, e) are given by the set of all permutations of each of $(10, 5, 5, 5, 5)$, $(2, 7, 7, 7, 7)$ or $(9, 3, 7, 5, 6)$.