The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

2010 Australian Intermediate Mathematics Olympiad Problems with some Solutions

1. A 3-digit number has the following properties.

   - If its tens and units digits are interchanged, its value increases by 36.
   - If its hundreds and units digits are interchanged, its value decreases by 198.

   When its hundreds and tens digits are interchanged, its value decreases.
   By how much?

   Solution. Let the 3-digit number be $N = (abc)$, i.e.

   $$N = 100a + 10b + c.$$

   Then, from the given information,

   $$\begin{aligned} & 10b + c + 36 = 10c + b \implies 9c - 9b = 36 && (1)\\ & 100a + c - 198 = 100c + a \implies 99a - 99c = 198 && (2)\\ \tfrac{1}{9}\cdot(1): \quad & c - b = 4 && (3)\\ \tfrac{1}{99}\cdot(2): \quad & a - c = 2 && (4)\\ \therefore (3)+(4): \quad & a - b = 6. \end{aligned}$$

   We want

   $$\begin{aligned} (100a + 10b) - (100b + a) &= 90(a - b)\\ &= 90 \cdot 6\\ &= 540. \end{aligned}$$

   Thus, when the number's hundreds and tens digits are interchanged, its value decreases by 540.

2. A number in base twelve is 3140. The same number in base b is 320.
   What is b?

   Solution. From the given information, we have:

   $$3 \times 12^3 + 1 \times 12^2 + 4 \times 12 = 3b^2 + 2b.$$

   Since 3 divides all terms other than $2b$, we have $3 \mid b$.
   $\therefore b = 3c$, for some $c \in \mathbb{N}$, and so dividing through by 3 we have

   $$12^3 + 4 \times 12 + 4 \times 4 = (3c)^2 + 2c.$$

Now 2 divides all terms other than $(3c)^2 = 9c^2$. Hence $2 \mid c$.
$\therefore c = 2d \implies b = 6d$, for some $d \in \mathbb{N}$, and so dividing through by 4 we have

$$\begin{aligned} 2 \times 6^3 + 12 + 4 &= 9d^2 + d \\ \therefore 1 &\equiv d \pmod 3 \end{aligned}$$

$\therefore d = 3k + 1$, for some $k \in \mathbb{N}$. So, $b = 18k + 6$.

$$\begin{aligned} 2 \times 6^3 + 12 + 3 + 1 &= 9(3k+1)^2 + 3k + 1 \\ 2 \times 6^3 + 12 + 3 &= 9(3k+1)^2 + 3k \\ 4 \times 6^2 + 4 + 1 &= 3(3k+1)^2 + k \\ &= 3(9k_6^2k + 1) + k \\ &= 27k^2 + 19k + 3 \\ \therefore 27k^2 + 19k - 146 &= 0. \end{aligned}$$

Now observe that $k = 2$ is a solution (and any other solution for k is negative). Thus

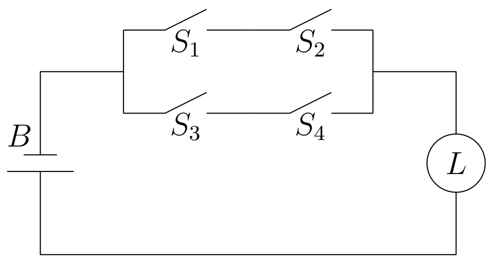
$$b = 18 \cdot 2 + 6 = 42.$$

3. Starship Conquest is pursuing starship Anarky, which is 12 klongs ahead of Conquest's current position. After Conquest has travelled 45 klongs, Anarky is just 7 klongs ahead.

   Assuming both starships are travelling on the same straight line at constant speeds, how many more klongs will it take for Conquest to catch Anarky?

   Solution. It takes 45 klongs to close 5 klongs.
   $\therefore$ Conquest takes 9 klongs to close 1 klong.
   $\therefore$ Conquest takes 63 klongs to close the remaining 7 klongs and hence catch Anarky.

4. In the diagram, B is a battery, L is a lamp, and S_1, S_2, S_3, S_4 are switches. The probability for each switch being on is $\frac{1}{3}$ and these probabilities are independent. Let p be the probability that the lamp is on.
   Find $729p$.



   Solution.

$$\begin{aligned} \Pr(S_1 \text{ and } S_2 \text{ on}) &= \tfrac{1}{3} \cdot \tfrac{1}{3} = \tfrac{1}{9} \\ &= \Pr(S_3 \text{ and } S_4 \text{ on}) \\ \Pr(\text{lamp is off}) &= \Pr(S_1 \text{ and } S_2 \text{ off}) \cdot \Pr(S_3 \text{ and } S_4 \text{ off}) \\ &= (1 - \tfrac{1}{9})^2 \\ p = \Pr(\text{lamp is on}) &= 1 - (1 - \tfrac{1}{9})^2 \\ \therefore 729p &= 9 \cdot \big(9^2 - (9-1)^2\big) \\ &= 9 \cdot (9 + 9 - 1)(1) \\ &= 9 \cdot 17 \\ &= 153. \end{aligned}$$

5. Jess has five bank accounts. If three account balances at a time were added, the following amounts would result: $94, 97, 99, 100, 101, 103, 104, 106, 107, 109$.

   What is the sum of the lowest and highest balances?

Solution. Suppose two of the accounts x, y have the same balance, and u, v are two other balances. Then $x + u + v = y + u + v$. However, all the triple sums given are different. So Jess's five bank accounts have different balances.
Let the bank balances be

$$b_1 < b_2 < b_3 < b_4 < b_5.$$

Summing the triple sums we have a total of $10 \cdot 3$ balances, and since each balance occurs an equal number of times, each balance must occur $\frac{1}{5}(10 \cdot 3) = 6$ times. So,

$$\begin{aligned}
6\sum_{i=1}^{5} b_i &= 94 + 97 + 99 + 100 + 101 + 103 + 104 + 106 + 107 + 109 \\
&= 1000 - 6 - 3 - 1 + 0 + 1 + 3 + 4 + 6 + 7 + 9 \\
&= 1020 \\
\therefore \sum_{i=1}^{5} b_i &= \tfrac{1}{6} \cdot 1020 = 170 \qquad (5)
\end{aligned}$$

So the balances taken two at a time are $170 - 109, \dots$, i.e.

$$61, 63, 64, 66, 67, 69, 70, 71, 73, 76.$$

In order the double sums are

$$b_1 + b_2 < b_1 + b_3 < \left\{\begin{matrix} b_2 + b_3 < b_2 + b_4 < b_3 + b_4 \\ b_1 + b_4 < b_1 + b_5 < b_2 + b_5 \end{matrix}\right\} < b_3 + b_5 < b_4 + b_5.$$

So we have

$$\begin{aligned}
b_1 + b_2 &= 61 \qquad (6) \\
b_1 + b_3 &= 63 \qquad (7) \\
b_3 + b_5 &= 73 \qquad (8) \\
b_4 + b_5 &= 76 \qquad (9)
\end{aligned}$$

and must determine $b_1 + b_5$:

$$\begin{aligned}
(5) - (6) - (9): \qquad b_3 &= \sum_{i=1}^{5} b_i - (b_1 + b_2) - (b_4 + b_5) \\
&= 170 - 61 - 76 \\
&= 33 \qquad (10) \\
(7) + (8) - 2 \cdot (10): \qquad b_1 + b_5 &= (b_1 + b_3) + (b_3 + b_5) - 2b_3 \\
&= 63 + 73 - 2 \cdot 33 \\
&= 30 + 40 = 70.
\end{aligned}$$

So the sum of the highest and lowest balances is 70.

6. Rectangle $ABCD$ has side lengths $AB = 195$ and $BC = 130$. Point P lies in its interior such that $CP = 117$ and $DP = 156$.

   Determine the length of AP.

7. Suppose $\triangle ABC$ has area 2010, and let X, Y and Z be points on the sides AB, BC and CA, respectively, such that

   $$\frac{AX}{XB} = \frac{2}{3}, \quad \frac{BY}{YC} = \frac{32}{35}, \quad \frac{CZ}{ZA} = \frac{1}{5}.$$

   Determine the area of $\triangle XYZ$.

8. If s is the smallest positive integer with the property that its digit sum and the digits sum of $s + 1$ are both divisible by 19, how many digits does s have?

9. Quadrilateral $ABCD$ is such that the midpoint O of AB is the centre of a circle to which AD, DC and CB are tangents; AB is not a diameter of the circle.

   Prove $AB^2 = 4AD \times BC$.

   Solution. We give a sketch of how one can find the solution.
   Draw a diagram: draw radii from O to the points of contact of the circle with sides AD, DC, CB of $ABCD$; let the points of contact with these sides be E, F, G, respectively.
   There are three pairs of congruent triangles: $\triangle DEO$ with $\triangle DFO$, and $\triangle CFO$ with $\triangle CGO$ are congruent pairs (triangles are formed by drawing tangents to a circle from a common point); less easy to see, is that $\triangle AEO$ and $\triangle BGO$ are also congruent, by the RHS Rule.
   Label all the angles about O noting which are equal via the above congruences; they sum to $180°$.
   Label the two equal angles at D, the two equal angles at C, and the angles at A and B; they sum to $360°$.
   Now look at the form of the result. How can such an equation arise? If it occurs to you that except for the 4, such equations may result from similar triangles, you're in business. Then observe that $\frac{1}{2}AB = AO = BO$, since O is the midpoint of AB. This leads one to consider $\triangle DOA$ and $\triangle OCB$. Can these triangles be shown to be similar? (The AA Rule is all we need, and it looks as if the angle sum equations we found before will do the rest.)

10. A 2-digit number is called *productive* if it is the product of two single digit numbers.

    Find all 9-digit numbers in which all digits are different and each pair of neighbouring digits forms a productive number.

    Investigation

    Find all pairs of consecutive numbers, each consisting of 8 distinct digits and for which each pair of neighbouring digits forms a productive number.

    Solution. We give a sketch of how one can find the solution.
    For brevity we will refer to a 9-digit number with the required properties as a required N or just "an N".
    First list all the productive numbers: 2×5 is the smallest. Be systematic so that none is missed. Now organise the data: list the productive numbers in a table according to their beginning digit, and again, according to their trailing digit.

Our strategy will be to draw a tree diagram or several tree diagrams, where the roots of each tree are either the possible beginning digits of each N or the possible final digits of each N. However, if we are not very disciplined, each tree will grow horrendously, and we won't be able to solve the question in the given time. So let us see if we can identify some results from our organised data that will help us prune the tree quickly.
From our organised data, we see that no productive number begins with 0 or 9. So there cannot be a digit to the right of 0 or 9 if one or other of these digits is in an N. This means that 0 or 9 can only appear at the end of an N, and so also, only one of 0 or 9 can occur at a time in an N.
From our organised data, we also see that there is only one productive number that begins with 7, and only one that ends in 7. Since exactly one of 0 or 9 is not in an N, all other digits are present and so 7 is a digit of each N. In fact, 7 must either be preceded by 2 or succeeded by 2, which implies it can't be an interior digit; 7 is not the last digit (0 or 9 is); so it must be the first digit, and hence, in fact each N begins with 72.
One can also identify that 63 and 81 must be an interior pair of digits of each N, and if 9 occurs then such an N ends in 49. We now have enough information to ensure the tree (only one!) we generate with all the possible N is kept well pruned, with the following branches:

$$724815630, 725481630, 728145630, 728163540, 728163549.$$

So these five numbers are the possible N, required.