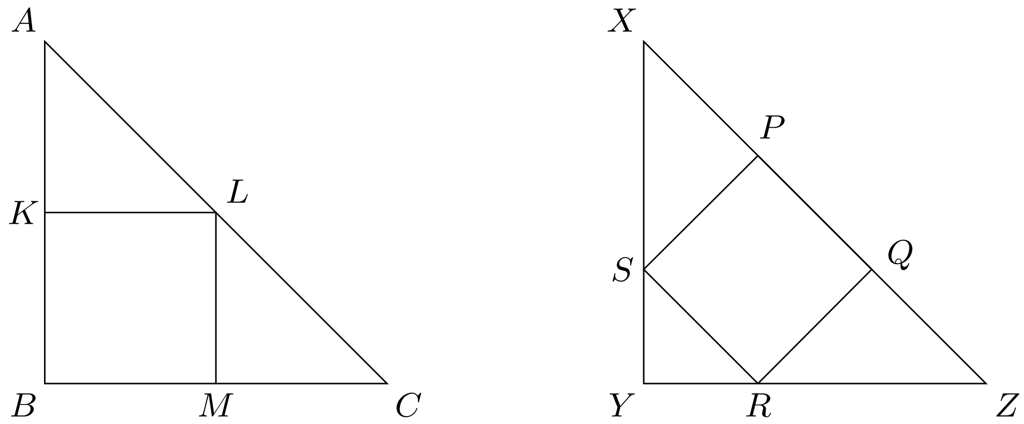
Time allowed: 4 hours. NO calculators are to be used.

Questions 1 to 8 only require their numerical answers all of which are non-negative integers less than 1000.
Questions 9 and 10 require written solutions which may include proofs.
The bonus marks for the Investigation in Question 10 may be used to determine prize winners.

1. In base b, the square of 24_b is 521_b. Find the value of b in base 10.
[**2 marks**]

2. Triangles ABC and XYZ are congruent right-angled isosceles triangles. Squares $KLMB$ and $PQRS$ are as shown. If the area of $KLMB$ is 189, find the area of $PQRS$.



[**2 marks**]

3. Let x and y be positive integers that simultaneously satisfy the equations $xy = 2048$ and $\frac{x}{y} - \frac{y}{x} = 7.875$. Find x. [**3 marks**]

4. Joel has a number of blocks, all with integer weight in kilograms. All the blocks of one colour have the same weight and blocks of a different colour have different weights.

Joel finds that various collections of some of these blocks have the same total weight w kg. These collections include:

1. 5 red, 3 blue and 5 green;
2. 4 red, 5 blue and 4 green;
3. 7 red, 4 blue and some green.

If $30 < w < 50$, what is the total weight in kilograms of 6 red, 7 blue and 3 green blocks? [**3 marks**]

5. Let $\frac{1}{a} + \frac{1}{b} = \frac{1}{20}$, where a and b are positive integers. Find the largest value of $a + b$. [**4 marks**]

6. Justin's sock drawer contains only identical black socks and identical white socks, a total of less than 50 socks altogether.

If he withdraws two socks at random, the probability that he gets a pair of the same colour is 0.5. What is the largest number of black socks he can have in his drawer? [**4 marks**]

7. A *code* is a sequence of 0s and 1s that does not have three consecutive 0s. Determine the number of codes that have exactly 11 digits. [**4 marks**]

8. Determine the largest integer n which has at most three digits and equals the remainder when n^2 is divided by 1000. [**4 marks**]

9. Let $ABCD$ be a trapezium with $AB \parallel CD$ such that

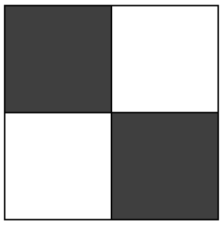
(i) its vertices A, B, C, D, lie on a circle with centre O,

(ii) its diagonals AC and BD intersect at point M and $\angle AMD = 60°$,

(iii) $MO = 10$.

Find the difference between the lengths of AB and CD. [**5 marks**]

10. An $n \times n$ grid with $n > 1$ is covered by several copies of a 2×2 square tile as in the figure below. Each tile covers precisely four cells of the grid and each cell of the grid is covered by at least one cell of one tile. The tiles may be rotated 90 degrees.



(a) Show there exists a covering of the grid such that there are exactly n black cells visible.
(b) Prove there is no covering where there are less than n black cells visible.
(c) Determine the maximum number of visible black cells. [**4 marks**]

Investigation

(i) Show that, for each possible pattern of 3 black cells and 6 white cells on a 3×3 grid, there is a covering whose visible cells have that pattern. [**1 bonus mark**]

(ii) Explain why not all patterns of 4 black cells and 12 white cells on a 4×4 grid can be achieved with a covering in which each new tile must be placed on top of all previous tiles that it overlaps. [**1 bonus mark**]

(iii) Determine the maximum number of visible black cells for a covering of an $n \times m$ grid where $1 < n < m$. [**2 bonus marks**]

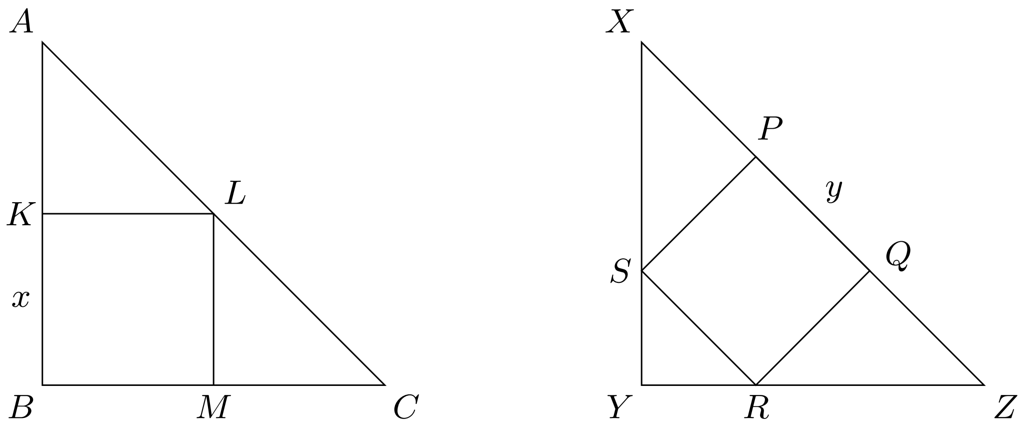
AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD SOLUTIONS

1. We have $24_b = 2b + 4$, $521_b = 5b^2 + 2b + 1$ and
$521_b = (2b + 4)^2 = 4b^2 + 16b + 16$.

Hence $0 = b^2 - 14b - 15 = (b - 15)(b + 1)$. Therefore $b = \mathbf{15}$.

2. *Preamble for Methods 1, 2, 3*

Let $BK = x$ and $PQ = y$.



Since ABC is a right-angled isosceles triangle and $BMLK$ is a square, CML and AKL are also right-angled isosceles triangles. Therefore $AK = CM = x$.

Since XYZ is a right-angled isosceles triangle and $PQRS$ is a square, XPS and ZQR and therefore YRS are also right-angled isosceles triangles. Therefore $XP = ZQ = y$.

Method 1

We have $3y = XZ = AC = AB\sqrt{2} = 2x\sqrt{2}$. So $y = \frac{2\sqrt{2}}{3}x$.
Hence the area of $PQRS = y^2 = \frac{8}{9}x^2 = \frac{8}{9} \times 189 = \mathbf{168}$.

Method 2

We have $2x = AB = \frac{AC}{\sqrt{2}} = \frac{XZ}{\sqrt{2}} = \frac{3y}{\sqrt{2}}$. So $y = \frac{2\sqrt{2}}{3}x$.
Hence the area of $PQRS = y^2 = \frac{8}{9}x^2 = \frac{8}{9} \times 189 = \mathbf{168}$.

Method 3

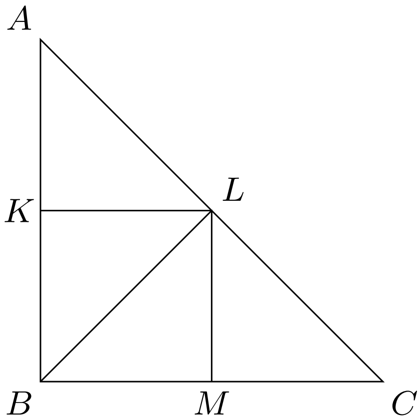
We have $2x = AB = XY = XS + SY = \sqrt{2}y + \frac{1}{\sqrt{2}}y = (\sqrt{2} + \frac{1}{\sqrt{2}})y = \frac{3}{\sqrt{2}}y$. So $y = \frac{2\sqrt{2}}{3}x$.

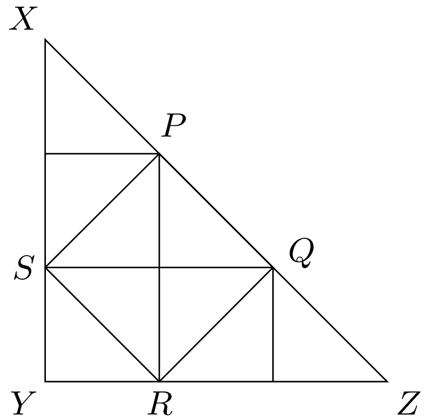
Hence the area of $PQRS = y^2 = \frac{8}{9}x^2 = \frac{8}{9} \times 189 = \mathbf{168}$.

Method 4

Joining B to L divides $\triangle ABC$ into 4 congruent right-angled isosceles triangles. Hence the area of $\triangle ABC$ is twice the area of $KLMB$.

Drawing the diagonals of $PQRS$ and the perpendiculars from P to XS and from Q to RZ divides $\triangle XYZ$ into 9 congruent right-angled isosceles triangles.





Hence the area of $PQRS = \frac{4}{9} \times$area of $\triangle XYZ = \frac{4}{9} \times$area of $\triangle ABC = \frac{4}{9} \times 2 \times$area of $KLMB = \frac{8}{9} \times 189 = \mathbf{168}$.

3. *Preamble for Methods 1, 2, 3*

Since x, y, and $\frac{x}{y} - \frac{y}{x}$ are all positive, we know that $x > y$. Since $xy = 2048 = 2^{11}$ and x and y are integers, we know that x and y are both powers of 2.

Method 1

Therefore (x,y) = (2048,1), (1024,2), (512,4), (256,8), (128,16), or (64,32).

Only (128,16) satisfies $\frac{x}{y} - \frac{y}{x} = 7.875 = 7\frac{7}{8} = \frac{63}{8}$. So $x = \mathbf{128}$.

Method 2

Let $x = 2^m$ and $y = 2^n$. Then $m > n$ and $xy = 2^{m+n}$, so $m+n = 11$.
From $\frac{x}{y} - \frac{y}{x} = 7.875 = 7\frac{7}{8}$ we have $2^{m-n} - 2^{n-m} = \frac{63}{8}$.
Let $m - n = t$. Then $2^t - 2^{-t} = \frac{63}{8}$.
So $0 = 8(2^t)^2 - 63(2^t) - 8 = (2^t - 8)(8(2^t) + 1)$.
Hence $2^t = 8 = 2^3$, $m - n = 3$, $2m = 14$, and $m = 7$.
Therefore $x = 2^7 = \mathbf{128}$.

Method 3

Let $x = 2^m$ and $y = 2^n$. Then $m > n$.
From $\frac{x}{y} - \frac{y}{x} = 7.875 = 7\frac{7}{8}$ we have
$x^2 - y^2 = \frac{63}{8}xy = \frac{63}{8} \times 2048 = 63 \times 2^8$.
So $63 \times 2^8 = (x - y)(x + y) = (2^m - 2^n)(2^m + 2^n) = 2^{2n}(2^{m-n} - 1)(2^{m-n} + 1)$.
Hence $2^{2n} = 2^8$, $2^{m-n} - 1 = 7$, and $2^{m-n} + 1 = 9$.
Therefore $n = 4$, $2^{m-4} = 8$, and $x = 2^m = 8 \times 2^4 = 2^7 = \mathbf{128}$.

Method 4

Now $\frac{x}{y} - \frac{y}{x} = 7.875 = 7\frac{7}{8} = \frac{63}{8}$ and $\frac{x}{y} - \frac{y}{x} = \frac{x^2-y^2}{xy} = \frac{x^2-y^2}{2048}$.
So $x^2 - y^2 = \frac{63}{8} \times 2048 = 63 \times 2^8 = (64 - 1)2^8 = (2^6 - 1)2^8 = 2^{14} - 2^8$.
Substituting $y = 2048/x = 2^{11}/x$ gives $x^2 - 2^{22}/x^2 = 2^{14} - 2^8$.
Hence $0 = (x^2)^2 - (2^{14} - 2^8)x^2 - 2^{22} = (x^2 - 2^{14})(x^2 + 2^8)$.
So $x^2 = 2^{14}$. Since x is positive, $x = 2^7 = \mathbf{128}$.

Method 5

We have $\frac{x}{y} - \frac{y}{x} = 7.875 = 7\frac{7}{8} = \frac{63}{8}$.
Multiplying by xy gives $x^2 - y^2 = \frac{63}{8}xy$.
So $8x^2 - 63xy - 8y^2 = 0$ and $(8x + y)(x - 8y) = 0$.
Since x and y are positive, $x = 8y$, $8y^2 = xy = 2048$, $y^2 = 256$, $y = 16$, $x = \mathbf{128}$.

Comment. From Method 4 or 5, we don't need to know that x and y are integers to solve this problem.

4. Let the red, blue and green blocks have different weights r, b and g kg respectively.
Then we have:

$$\begin{aligned} 5r + 3b + 5g &= w \qquad (1) \\ 4r + 5b + 4g &= w \qquad (2) \\ 7r + 4b + ng &= w \qquad (3) \end{aligned}$$

where n is the number of green blocks.

Subtracting (1) and (2) gives $2b = r + g$.
Substituting in (2) gives $13b = w$, so w is a multiple of 13 between 30 and 50.
Hence $w = 39$, $b = 3$ and $r + g = 6$.

Method 1

Since $r + g = 6$, r is one of the numbers 1, 2, 4, 5.
If r is 4 or 5, $7r + 4b > 39$ and (3) cannot be satisfied.
If $r = 2$, then $g = 4$ and (3) gives $26 + 4n = 39$, which cannot be satisfied in integers.
So $r = 1$, then $g = 5$ and (3) gives $19 + 5n = 39$ and $n = 4$.
Hence the total weight in kilograms of 6 red, 7 blue, and 3 green blocks is $6 \times 1 + 7 \times 3 + 3 \times 5 = \mathbf{42}$.

Method 2

Since $r + g = 6$, g is one of the numbers 1, 2, 4, 5.
Substituting $r = 6 - g$ in (3) gives $(7 - n)g = 15$. Thus g is 1 or 5.
If $g = 1$, then $n = -8$, which is not allowed.
If $g = 5$, then $n = 4$ and $r = 1$.
Hence the total weight in kilograms of 6 red, 7 blue, and 3 green blocks is $6 \times 1 + 7 \times 3 + 3 \times 5 = \mathbf{42}$.

5. *Method 1*

From symmetry we may assume $a \leq b$. If $a = b$, then both are 40 and $a + b = 80$. We now assume $a < b$. As a increases, b must decrease to satisfy the equation $\frac{1}{a} + \frac{1}{b} = \frac{1}{20}$. So $a < 40$.
We have $\frac{1}{b} = \frac{1}{20} - \frac{1}{a} = \frac{a-20}{20a}$. So $b = \frac{20a}{a-20}$. Since a and b are positive, $a > 20$.

The table shows all integer values of a and b.

a	21	22	24	25	28	30	36
b	420	220	120	100	70	60	45

Thus the largest value of $a + b$ is $21 + 420 =$ **441**.

Method 2

As in Method 1, we have $b = \frac{20a}{a-20}$ and $20 < a \leq 40$.

So $a + b = a(1 + \frac{20}{a-20})$.
If $a = 21$, than $a + b = 21(1 + 20) = 441$.
If $a \geq 22$, then $a + b \leq 40(1 + 10) = 440$.

Thus the largest value of $a + b$ is **441**.

Method 3

We have $ab = 20(a + b)$. So $(a - 20)(b - 20) = 400 = 2^4 5^2$.

Since b is positive, $ab > 20a$ and $a > 20$. Similarly $b > 20$.

From symmetry we may assume $a \leq b$ hence $a - 20 \leq b - 20$.

The table shows all values of $a - 20$ and the corresponding values of $b - 20$.

$a - 20$	1	2	4	8	16	5	10	20
$b - 20$	400	200	100	50	25	80	40	20

Thus the largest value of $a + b$ is $21 + 420 =$ **441**.

Method 4

We have $ab = 20(a + b)$, so 5 divides a or b. Since b is positive, $ab > 20a$ and $a > 20$.

Suppose 5 divides a and b. From symmetry we may assume $a \leq b$. The following table gives all values of a and b.

a	25	30	40
b	100	60	40

Suppose 5 divides a but not b. Since $b(a - 20) = 20a$, 25 divides $a - 20$. Let $a = 20 + 25n$. Then $(20 + 25n)b = 20(20 + 25n + b)$,

$nb = 16 + 20n$, $n(b - 20) = 16$. The following table gives all values of n, b, and a.

n	1	2	4	8	16
$b-20$	16	8	4	2	1
b	36	28	24	22	21
a	45	70	120	220	420

A similar table is obtained if 5 divides b but not a.

Thus the largest value of $a + b$ is $21 + 420 =$ **441**.

Method 5

We have $ab = 20(a + b)$. So maximising $a + b$ is equivalent to maximising ab, which is equivalent to minimising $\frac{1}{ab}$.

Let $x = \frac{1}{a}$ and $y = \frac{1}{b}$. We want to minimise xy subject to $x + y = \frac{1}{20}$. From symmetry we may assume $x \geq y$. Hence $x \geq \frac{1}{40}$.

Thus we want to minimise $z = x(\frac{1}{20} - x)$ with $z > 0$, hence with $0 < x < \frac{1}{20}$. The graph of this function is an inverted parabola with its turning point at $x = \frac{1}{40}$. So the minimum occurs at $x = \frac{1}{21}$. This corresponds to $y = \frac{1}{20} - \frac{1}{21} = \frac{1}{420}$.

Thus the largest value of $a + b$ is $21 + 420 =$ **441**.

6. *Method 1*

Let b be the number of black socks and w the number of white ones. If b or w is 0, then the probability of withdrawing a pair of socks of the same colour would be 1. So b and w are positive. From symmetry we may assume that $b \geq w$.

The number of pairs of black socks is $b(b-1)/2$. The number of pairs of white socks is $w(w-1)/2$. The number of pairs of socks with one black and the other white is bw.

The probability of selecting a pair of socks of the same colour is the same as the probability of selecting a pair of socks of different colour. Hence $b(b-1)/2 + w(w-1)/2 = bw$ or

$$b(b-1) + w(w-1) = 2bw$$

.

Let $d = b - w$. Then $w = b - d$ and

$$\begin{aligned} b(b-1) + (b-d)(b-d-1) &= 2b(b-d) \\ b^2 - b + b^2 - bd - b - bd + d^2 + d &= 2b^2 - 2bd \\ -2b + d^2 + d &= 0 \\ d(d+1) &= 2b \end{aligned}$$

The following table shows all possible values of d.
Note that $b + w = 2b - d = d^2$.

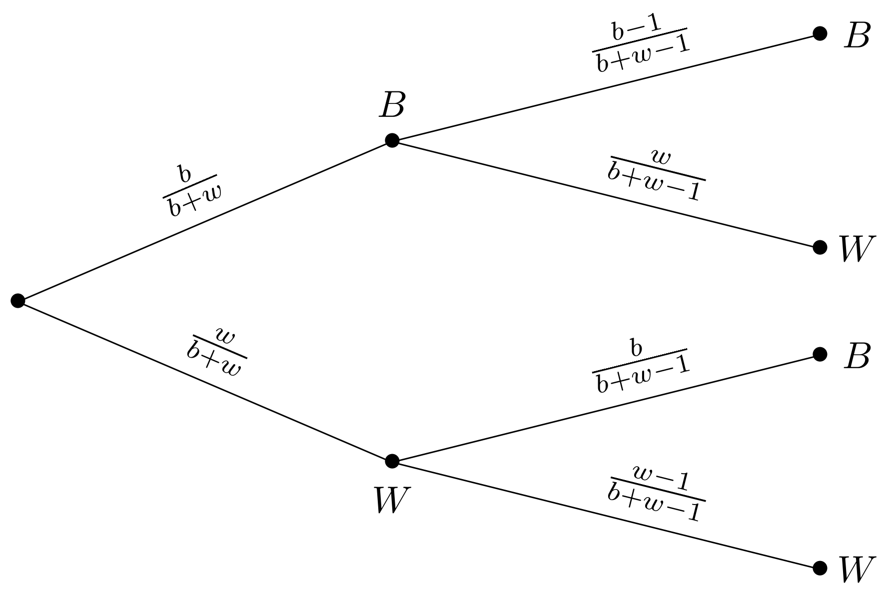
d	0	1	2	3	4	5	6	7	≥ 8
b	0	1	3	6	10	15	21	28	
$b+w$	0	1	4	9	16	25	36	49	≥ 64

Thus the largest value of b is **28**.

Preamble for Methods 2, 3, 4

Let b be the number of black socks and w the number of white ones. If b or w is 0, then the probability of withdrawing a pair of socks of the same colour would be 1. So b and w are positive. From symmetry we may assume that $b \geq w$.

The pair of socks that Justin withdraws are either the same colour or different colours. So the probability that he draws a pair of socks of different colours is $1 - 0.5 = 0.5$. The following diagram shows the probabilities of withdrawing one sock at a time.



So the probability that Justin draws a pair of socks of different colours is $\frac{2bw}{(b+w)(b+w-1)}$.
Hence $4bw = b^2 + 2bw + w^2 - b - w$ and $b^2 - 2bw + w^2 - b - w = 0$.

Method 2

We have $b^2 - (2w+1)b + (w^2 - w) = 0$.
The quadratic formula gives
$b = (2w + 1 \pm \sqrt{(2w+1)^2 - 4(w^2 - w)})/2 = (2w + 1 \pm \sqrt{8w+1})/2$.
If $b = (2w + 1 - \sqrt{8w+1})/2 = w + \frac{1}{2} - \frac{1}{2}\sqrt{8w+1}$,
then $b \leq w + \frac{1}{2} - \frac{1}{2}\sqrt{9} = w - 1 < w$.
So $b = (2w + 1 + \sqrt{8w+1})/2$.

Now $w < 25$ otherwise $b + w \geq 2w \geq 50$.
Since b increases with w, we want the largest value of w for which $8w + 1$ is square. Thus $w = 21$ and the largest value of b is
$(42 + 1 + \sqrt{169})/2 = (43 + 13)/2 = \mathbf{28}$.

Method 3

We have $b + w = (b - w)^2$. Thus $b + w$ is a square number less than 50 and greater than 1.

The following tables gives all values of $b + w$ and the corresponding values of $b - w$ and b.

$b+w$	4	9	16	25	36	49
$b-w$	2	3	4	5	6	7
$2b$	6	12	20	30	42	56
b	3	6	10	15	21	28

Thus the largest value of b is **28**.

Method 4

We have $b + w = (b - w)^2$. Also $w < 25$ otherwise $b + w \geq 2w \geq 50$.
For a fixed value of w, consider the line $y = w + b$ and parabola $y = (b - w)^2$. These intersect at a unique point for $b \geq w$. For each value of w we guess and check a value of b for which the line and parabola intersect.

w	b	$b+w$	$(b-w)^2$	$b+w=(b-w)^2$?
24	31	55	49	$b+w>(b-w)^2$
	32	56	64	$b+w<(b-w)^2$
23	30	53	49	$b+w>(b-w)^2$
	31	54	64	$b+w<(b-w)^2$
22	29	51	49	$b+w>(b-w)^2$
	30	52	64	$b+w<(b-w)^2$
21	27	48	36	$b+w>(b-w)^2$
	28	49	49	$b+w=(b-w)^2$

As w decreases, the line $y = w + b$ shifts down and the parabola $y = (b - w)^2$ shifts left so their point of intersection shifts left. So b decreases as w decreases. Thus the largest value of b is **28**.

Comment. We have $b+w = (b-w)^2$. Let $b-w = n$. Then $b+w = n^2$. Hence $b = (n^2+n)/2 = n(n+1)/2$ and $w = (n^2-n)/2 = (n-1)n/2$. Thus w and b are consecutive triangular numbers.

7. *Method 1*

Let c_n be the number of codes that have exactly n digits.

For $n \geq 4$, a code with n digits ends with 1 or 10 or 100.

If the code ends in 1, then the string that remains when the end digit is removed is also a code. So the number of codes that end in 1 and have exactly n digits equals c_{n-1}.

If the code ends in 10, then the string that remains when the last 2 digits are removed is also a code. So the number of codes that end in 10 and have exactly n digits equals c_{n-2}.

If the code ends in 100, then the string that remains when the last 3 digits are removed is also a code. So the number of codes that end in 100 and have exactly n digits equals c_{n-3}.

Hence, for $n \geq 4$, $c_n = c_{n-1} + c_{n-2} + c_{n-3}$.

By direct counting, $c_1 = 2$, $c_2 = 4$, $c_3 = 7$. The table shows c_n for $1 \leq n \leq 11$.

n	1	2	3	4	5	6	7	8	9	10	11
c_n	2	4	7	13	24	44	81	149	274	504	927

Thus the number of codes that have exactly 11 digits is **927**.

Method 2

Let c_n be the number of codes that have exactly n digits.

A code ends with 0 or 1.

Suppose $n \geq 5$. If a code ends with 1, then the string that remains when the end digit is removed is also a code. So the number of codes that end with 1 and have exactly n digits equals c_{n-1}.

If a code with n digits ends in 0, then the string that remains when the end digit is removed is a code with $n-1$ digits that does not end with two 0s. If a code with $n-1$ digits ends with two 0s, then it ends with 100. If the 100 is removed then the string that remains is an unrestricted code that has exactly $n-4$ digits. So the number of codes with $n-1$ digits that do not end with two 0s is $c_{n-1} - c_{n-4}$.

Hence, for $n \geq 5$, $c_n = 2c_{n-1} - c_{n-4}$.

By direct counting, $c_1 = 2$, $c_2 = 4$, $c_3 = 7$, $c_4 = 13$. The table shows c_n for $1 \leq n \leq 11$.

n	1	2	3	4	5	6	7	8	9	10	11
c_n	2	4	7	13	24	44	81	149	274	504	927

Thus the number of codes that have exactly 11 digits is **927**.

Comment. The equation $c_n = 2c_{n-1} - c_{n-4}$ can also be derived from the equation $c_n = c_{n-1} + c_{n-2} + c_{n-3}$.
For $n \geq 5$ we have $c_{n-1} = c_{n-2} + c_{n-3} + c_{n-4}$.
Hence $c_n = c_{n-1} + (c_{n-1} - c_{n-4}) = 2c_{n-1} - c_{n-4}$.

Method 3

Let c_n be the number of codes that have exactly n digits.

By direct counting, $c_1 = 2$, $c_2 = 4$, $c_3 = 7$, $c_4 = 13$.

A code with exactly 5 digits has the form xx1xx or x101x or 1001x or x1001, where each x is a digit.
The number of codes of the form xx1xx is $4 \times 4 = 16$.
The number of codes of the form x101x is $2 \times 2 = 4$.
The number of codes of the form 1001x is 2.

The number of codes of the form x1001 is 2.
So $c_5 = 16 + 4 + 2 + 2 = 24$.

A code with exactly 11 digits has the form xxxxx1xxxxx or xxxx101xxxx or xxx1001xxxx or xxxx1001xxx, where each x is a digit.
The number of codes of the form xxxxx1xxxxx is $24 \times 24 = 576$.
The number of codes of the form xxxx101xxxx is $13 \times 13 = 169$.
The number of codes of the form xxx1001xxxx is $7 \times 13 = 91$.
The number of codes of the form xxxx1001xxx is $13 \times 7 = 91$.
So $c_{11} = 576 + 169 + 91 + 91 = \mathbf{927}$.

8. *Method 1*

The square of n has the same last three digits of n if and only if $n^2 - n = n(n-1)$ is divisible by $1000 = 2^3 \times 5^3$.

As n and $n-1$ are relatively prime, only one of those two numbers is even and only one of them can be divisible by 5. This yields the following cases.

Case 1. n is divisible by both 2^3 and 5^3.
Then $n \geq 1000$, a contradiction.

Case 2. $n-1$ is divisible by both 2^3 and 5^3.
Then $n \geq 1001$, a contradiction.

Case 3. n is divisible by 2^3 and $n-1$ is divisible by 5^3. The second condition implies that n is one of the numbers 1, 126, 251, 376, 501, 626, 751, 876. Since n is also divisible by 8, this leaves $n = 376$.

Case 4. n is divisible by 5^3 and $n-1$ is divisible by 2^3. The first condition implies that n is one of the numbers 125, 250, 375, 500, 625, 750, 875. But n must also leave remainder 1 when divided by 8, which leaves $n = 625$.

Therefore $n = \mathbf{625}$.

Method 2

We want a number n and its square to have the same last three digits.

First, n and n^2 should have the same last digit. Squaring each of the digits from 0 to 9 shows that the last digit of n must be 0, 1, 5 or 6.

Second, n and n^2 should have the same last two digits. Squaring each of the 2-digit numbers 00 to 90, 01 to 91, 05 to 95, and 06 to 96 as in the following table shows that the last two digits of n must be 00, 01, 25 or 76.

n	n^2	n	n^2	n	n^2	n	n^2
00	00	01	01	05	25	06	36
10	00	11	21	15	25	16	56
20	00	21	41	25	25	26	76
30	00	31	61	35	25	36	96
40	00	41	81	45	25	46	16
50	00	51	01	55	25	56	36
60	00	61	21	65	25	66	56
70	00	71	41	75	25	76	76
80	00	81	61	85	25	86	96
90	00	91	81	95	25	96	16

Finally, n and n^2 should have the same last three digits. Squaring each of the 3-digit numbers 000 to 900, 001 to 901, 025 to 925, and 076 to 976 as in the following table shows that the last three digits of n must be 000, 001, 625 or 376.

n	n^2	n	n^2	n	n^2	n	n^2
000	000	001	001	025	625	076	776
100	000	101	201	125	625	176	976
200	000	201	401	225	625	276	176
300	000	301	601	325	625	376	376
400	000	401	801	425	625	476	576
500	000	501	001	525	625	576	776
600	000	601	201	625	625	676	976
700	000	701	401	725	625	776	176
800	000	801	601	825	625	876	376
900	000	901	801	925	625	976	576

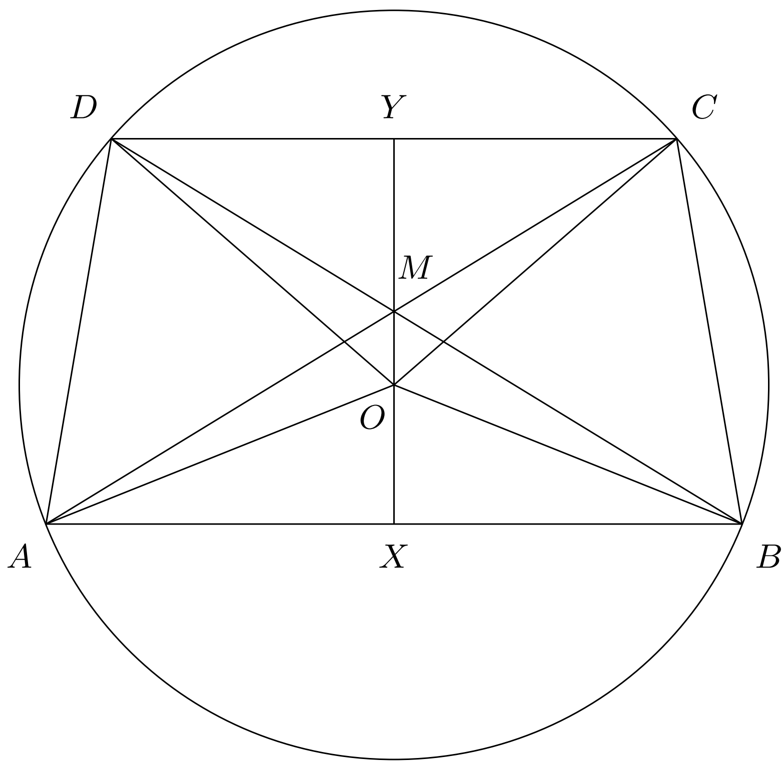
Therefore $n = \mathbf{625}$.

9. *Preamble*

Since $ABCD$ is a cyclic quadrilateral, $\angle DCA = \angle DBA$.
Since $AB \parallel CD$, $\angle DCA = \angle CAB$. So $\triangle AMB$ is isosceles.
Similarly $\triangle CMD$ is isosceles.

Extend MO to intersect AB at X and CD at Y.

Since $OA = OB$, triangles AMO and BMO are congruent. So $\angle AMO = \angle BMO$. Since $\angle AMD = 60°$, $\angle AMB = 120°$ and $\angle AMO = \angle BMO = 60°$. Hence triangles AMX and BMX are congruent and have angles $30°$, $60°$, $90°$. Similarly DMY and CMY are congruent 30-60-90 triangles.



Method 1

We know that X and Y are the midpoints of AB and CD respectively. Let $2x$ and $2y$ be the lengths of AB and CD respectively. From the 30-90-60 triangles AXM and CYM we have $XM = \frac{x}{\sqrt{3}}$ and $YM = \frac{y}{\sqrt{3}}$.

From the right-angled triangles AXO and CYO, Pythagoras gives

$$AO^2 = x^2 + (\frac{x}{\sqrt{3}} - 10)^2 = \frac{4}{3}x^2 + 100 - \frac{20}{\sqrt{3}}x$$

$$CO^2 = y^2 + (\frac{y}{\sqrt{3}} + 10)^2 = \frac{4}{3}y^2 + 100 + \frac{20}{\sqrt{3}}y$$

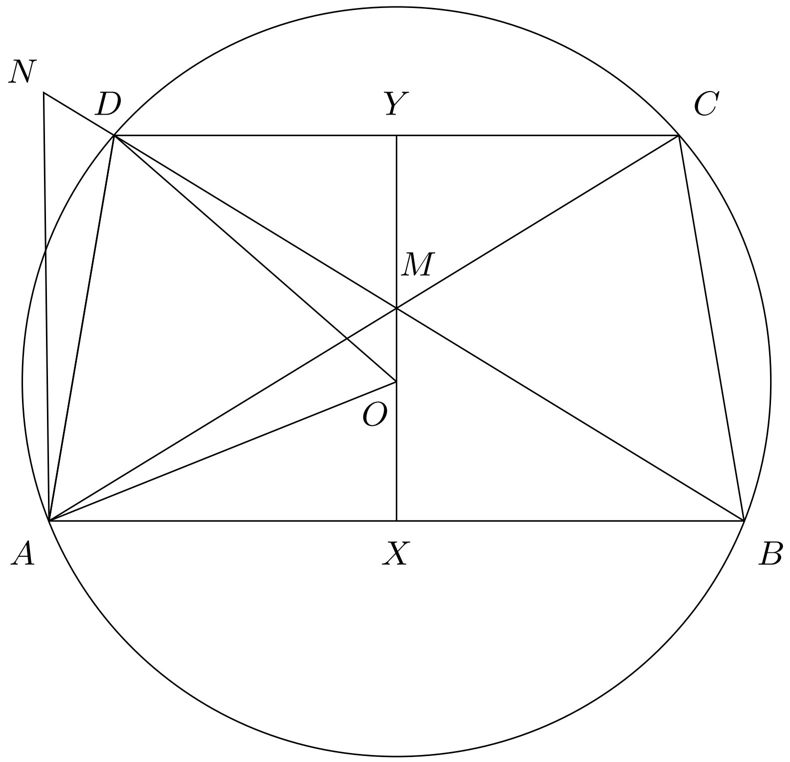
These equations also hold if O lies outside the trapezium $ABCD$.

Since $AO = CO$, we have
$\frac{4}{3}(x^2 - y^2) = \frac{20}{\sqrt{3}}(x + y)$, $x^2 - y^2 = 5\sqrt{3}(x + y)$, $x - y = 5\sqrt{3}$ and $AB - CD = 2(x - y) = 10\sqrt{3}$.

Method 2

We know that $\angle ABD = 30°$. Since O is the centre of the circle we have $\angle AOD = 2\angle ABD = 60°$. Thus $\angle AOD = \angle AMD$, hence $AOMD$ is cyclic. Since $OA = OD$ and $\angle AOD = 60°$, $\triangle AOD$ is equilateral.

Rotate $\triangle AOM$ $60°$ anticlockwise about A to form triangle ADN.



Since $AOMD$ is cyclic, $\angle AOM + \angle ADM = 180°$. Hence MDN is a straight line. Since $\angle AMD = 60°$ and $AM = AN$, $\triangle AMN$ is equilateral. So $AM = MN = MD + DN = MD + MO$.

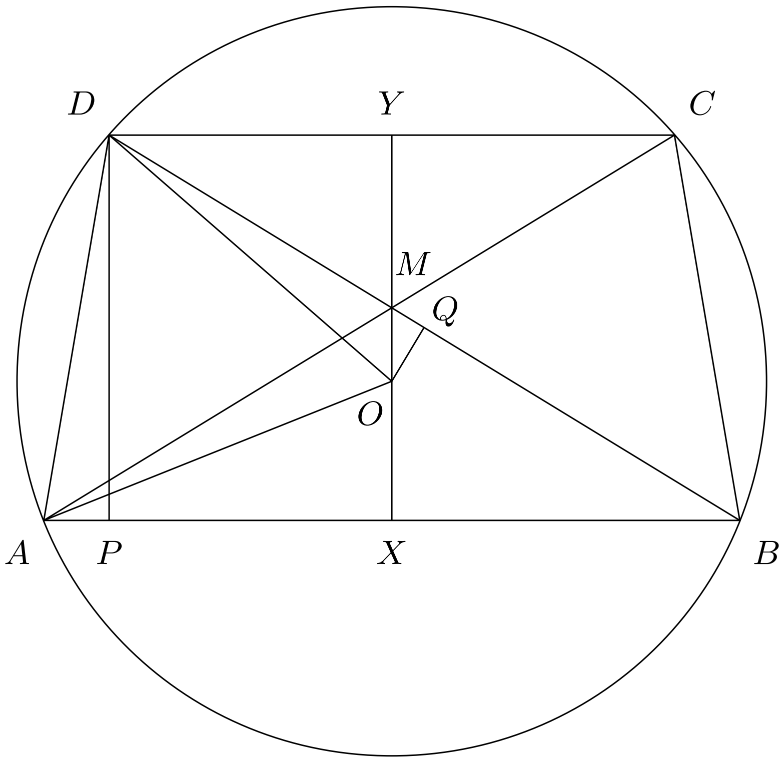
[Alternatively, applying Ptolemy's theorem to the cyclic quadrilateral $AOMD$ gives $AO \times MD + AD \times MO = AM \times OD$. Since $AO = AD = OD$, cancelling these gives $MD + MO = AM$.]

We know that X and Y are the midpoints of AB and CD respectively.
From the 30-90-60 triangles AXM and DYM we have $AX = \frac{\sqrt{3}}{2}AM$ and $DY = \frac{\sqrt{3}}{2}DM$.
So $AB - CD = 2(\frac{\sqrt{3}}{2}AM - \frac{\sqrt{3}}{2}DM) = \sqrt{3}MO = 10\sqrt{3}$.

Method 3

As in Method 2, $\triangle AOD$ is equilateral.

Let P and Q be points on AB and BD respectively so that $DP \perp AB$ and $OQ \perp BD$.



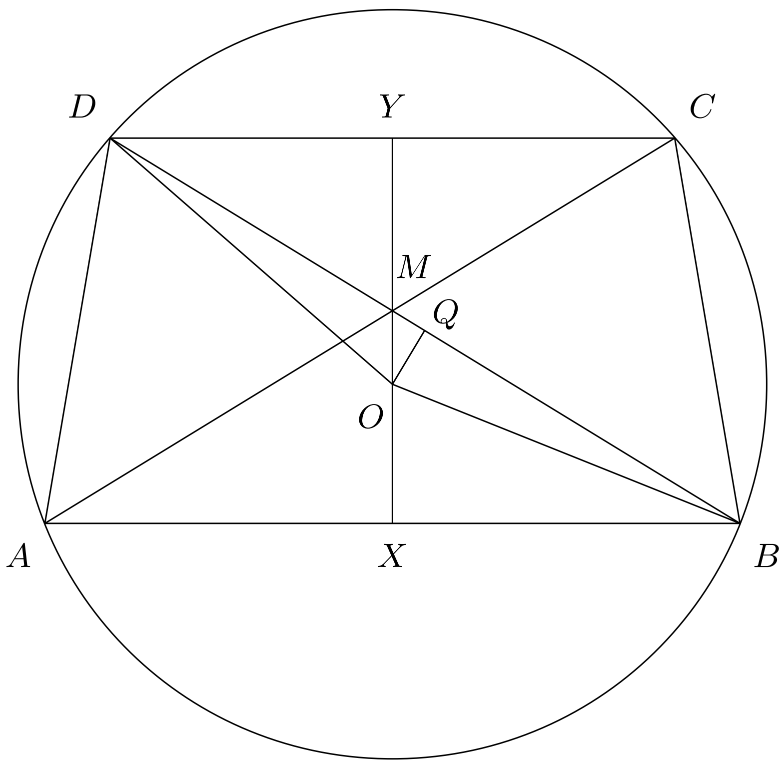
From the 30-60-90 triangle BDP, $DP = \frac{1}{2}BD$. Since $OB = OD$, triangles DOQ and BOQ are congruent. Hence $DQ = \frac{1}{2}DB = DP$. So triangles APD and OQD are congruent. Therefore $AP = OQ$.

From the 30-60-90 triangle OMQ, $OQ = \frac{\sqrt{3}}{2}OM = 5\sqrt{3}$.
So $AB - CD = 2AX - 2DY = 2AX - 2PX = 2AP = 10\sqrt{3}$.

Method 4

Let $x = BM$ and $y = DM$. From the 30-90-60 triangles BXM and DYM we have $BX = \frac{\sqrt{3}}{2}x$ and $DY = \frac{\sqrt{3}}{2}y$. Since X and Y are the midpoints of AB and CD respectively, $AB - CD = \sqrt{3}(x - y)$.

Let Q be the point on BD so that $OQ \perp BD$.

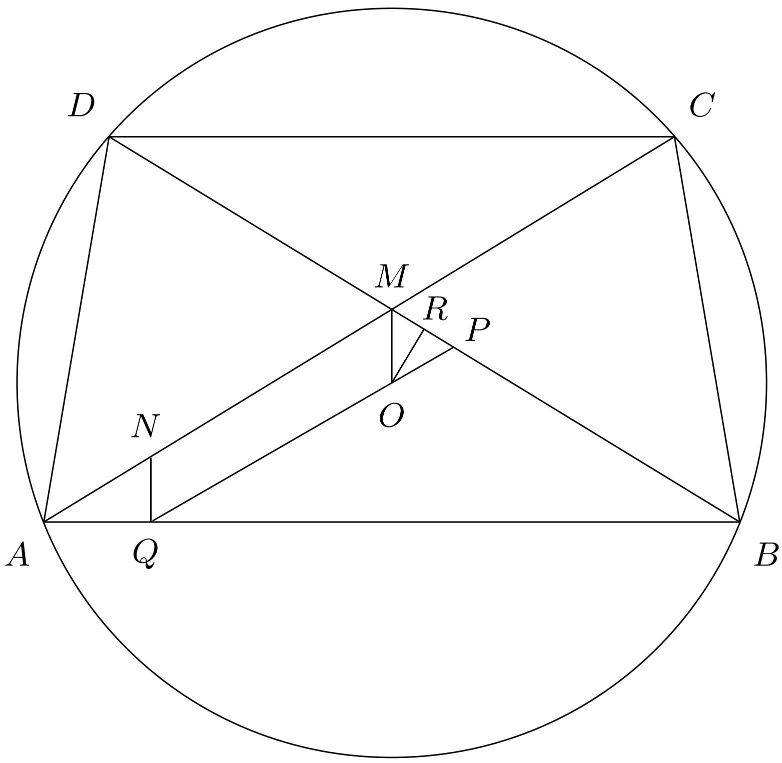


Since $BO = DO$, triangles BQO and DQO are congruent and $BQ = DQ$. Therefore $BQ = (x + y)/2$ and $MQ = x - BQ = (x - y)/2$.

Since BXM is a 30-90-60 triangle, $\triangle OQM$ is also 30-90-60. Therefore $MQ = \frac{1}{2}MO = 5$. So $AB - CD = 2\sqrt{3}MQ = 10\sqrt{3}$.

Method 5

We know that triangles AMB and DMC have the same angles. Let the line that passes through O and is parallel to AC intersect AB at Q and BD at P. Then $\angle BQP = \angle BAM$ and $\angle BPQ = \angle BMA$. So triangles BPQ and CMD are similar.



Now $\angle QPD = \angle AMD = 60°$. So $\triangle OMP$ is equilateral. Let the line that passes through O and is perpendicular to BD intersect BD at R. Thus R bisects PM. Since $OD = OB$, triangles OBR and ODR are congruent and R bisects BD. Hence $DM = BP$ and triangles BPQ and CMD are congruent. So $AB - CD = AQ$.

Draw QN parallel to OM with N on AM. Then $QN = OM = 10$ and $QN \perp AB$. So $\triangle ANQ$ is 30-60-90. Hence $AN = 20$ and, by Pythagoras, $AB - CD = \sqrt{400 - 100} = \sqrt{300} = 10\sqrt{3}$.

Comment. Notice that $AB - CD$ is independent of the radius of the circumcircle $ABCD$. This is true for all cyclic trapeziums. If $\angle AMD = \alpha$, then by similar arguments to those above we can show that $AB - CD = 2MO \sin\alpha$.

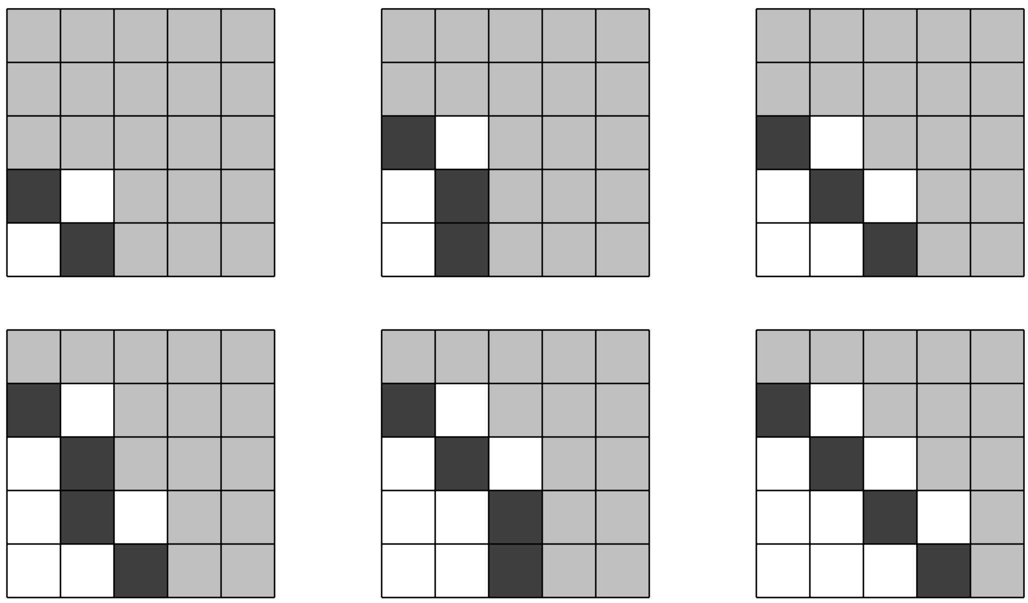
10. (a) Mark cells of the grid by coordinates, with (1, 1) being the cell in the lower-left corner of the grid. There are many ways of achieving a covering with exactly n black cells visible. Here's three.

Method 1

Putting each new tile *above* all previous tiles it overlaps with, place tiles in the following order with their lower-left cells on the listed grid cells:

(1, 1),
(1, 2), (2, 1),
(1, 3), (2, 2), (3, 1),
(1, 4), (2, 3), (3, 2), (4, 1),
and so on.

Continue this procedure to give black cells on the 'diagonal' just below the main diagonal and only white cells below. The following diagram shows this procedure for $n = 5$.



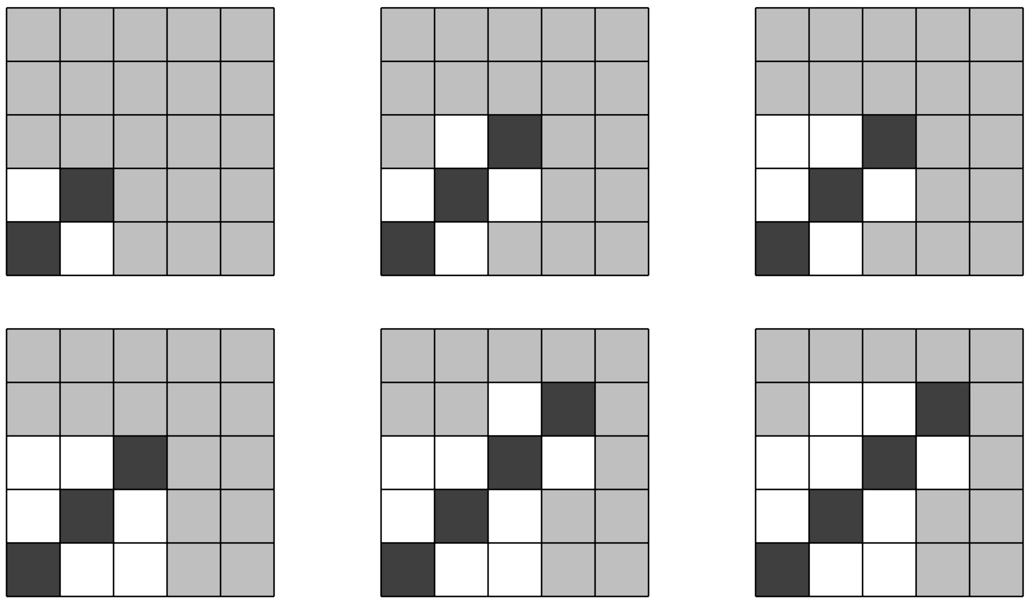
Start then in the upper-right corner and create black cells on the 'diagonal' just above the main diagonal and only white cells above. Finally put $n-1$ tiles along the main diagonal. That will give n black cells on the main diagonal and white cells everywhere else.

Method 2

Rotate all tiles so that the lower-left and upper-right cells are black. Putting each new tile *underneath* all previous tiles it overlaps with, place tiles in the following order with their lower-left cells on the listed grid cells:

(1, 1),
(2, 2), (1, 2), (2, 1),
(3, 3), (2, 3), (1, 3), (3, 2), (3, 1),
(4, 4), (3, 4), (2, 4), (1, 4), (4, 3), (4, 2), (4, 1),
and so on.

Continuing this procedure gives n black cells on the diagonal and white cells everywhere else. The following diagram shows this procedure for $n = 5$.

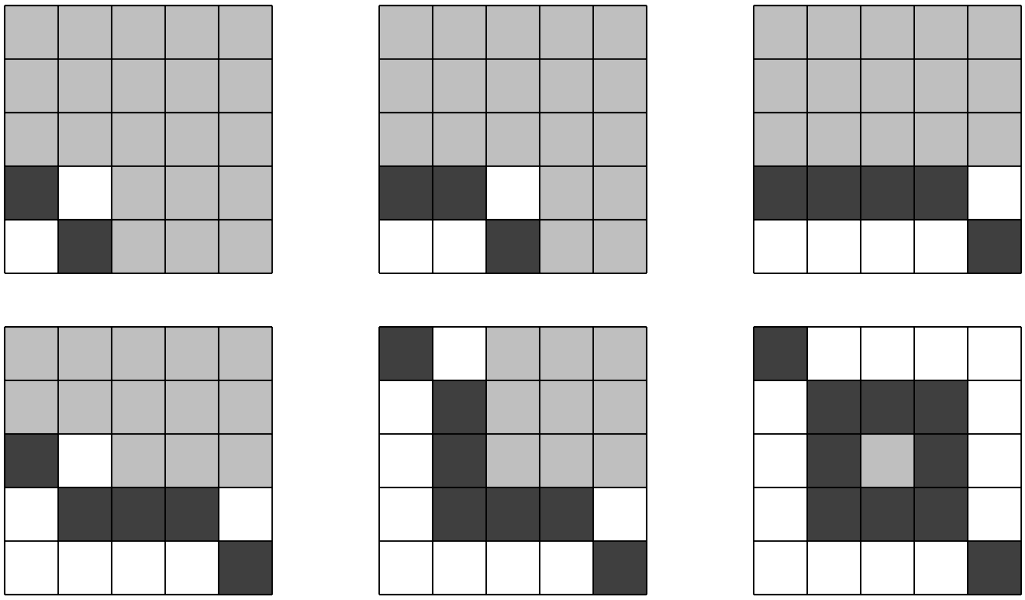


Method 3

Putting each new tile *above* all previous tiles it overlaps with, place tiles in the following order with their lower-left cells on the listed grid cells:

$(1, 1), (2, 1), (3, 1), \ldots, (n-1, 1),$
$(1, 2), (1, 2), (1, 3), \ldots, (1, n-1),$
$(n-1, n-1), (n-2, n-1), \ldots, (1, n-1),$
$(n-1, n-2), (n-1, n-3), \ldots, (n-1, 1),$

The following diagram shows this procedure for $n = 5$.



This gives a single border of all white cells except for black cells in the top-left and bottom-right corners of the grid. Now repeat this procedure for the inner $(n-2)\times(n-2)$ grid, then the inner $(n-4)\times(n-4)$ grid, and so on until an inner 1×1 or 2×2 grid remains. In both cases a single tile can cover the remaining uncovered grid cell(s) to produce a total covering that has n black cells on the diagonal and white cells everywhere else.

(b) Suppose there is a covering of the $n\times n$ grid that has less than n black cells visible. Then there must be a row in which all visible cells are white. Any tile that overlaps this row has exactly two cells that coincide with cells in the row. These two cells are in the same row of the tile so one is white and one is black. Call these two cells a half-tile. Consider all half-tiles that cover cells in the row. Remove any half-tiles that have neither cell visible. The remaining half-tiles cover the row and all their visible cells are white.

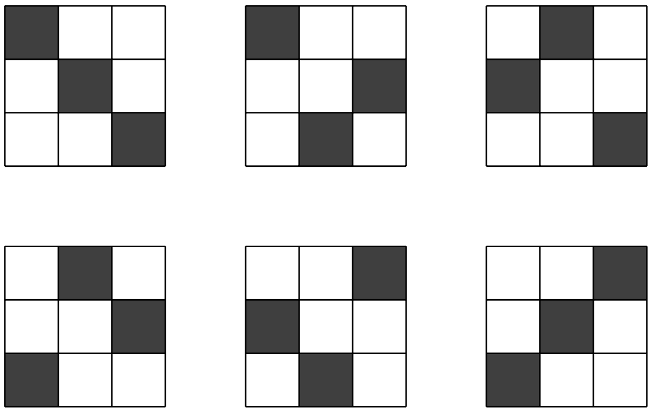
Consider any half-tile H_1. The black cell of H_1 must be covered by some half-tile H_2 and the white cell of H_1 must be visible. The black cell of H_2 must be covered by some half-tile H_3 and the white cell of H_2 must be visible. Thus we have a total of two visible white cells in the row. The black cell of H_3 must be covered by some half-tile H_4 and the white cell of H_3 must be visible. Thus we have a total of three visible white cells in the row.

So we may continue until we have a half-tile H_{n-1} plus a total of $n-2$ visible white cells in the row. The black cell of H_{n-1} must be covered by some half-tile H_n and the white cell of H_{n-1} must be visible. Thus we have a total of $n-1$ visible white cells in the row. As there are only n cells in the row, H_n must cover one of the visible white cells. This is a contradiction. So every covering of the $n \times n$ grid has at least n black cells visible.

(c) From (a) and (b), the minimum number of visible black cells is n. From symmetry, the minimum number of visible white cells is n. Hence the maximum number of visible black cells is $n^2 - n$.

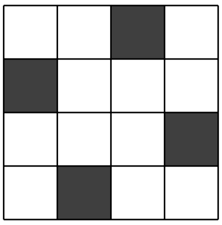
Investigation

(i) If a covering of a 3×3 grid has exactly 3 visible black cells, then the argument in Part (b) above shows that each row and each column must have exactly one visible black cell. The following diagram shows all possible patterns with exactly 3 black cells.



From symmetry we only need to consider the first two patterns. A covering to achieve the first pattern was given in Part (a) above. The second pattern can be achieved from the first by rotating a tile 90° and placing it in the bottom-right corner of the grid.

(ii) The last tile to be placed shows two visible black cells and they share a vertex. However, in the following pattern no two black cells share a vertex.

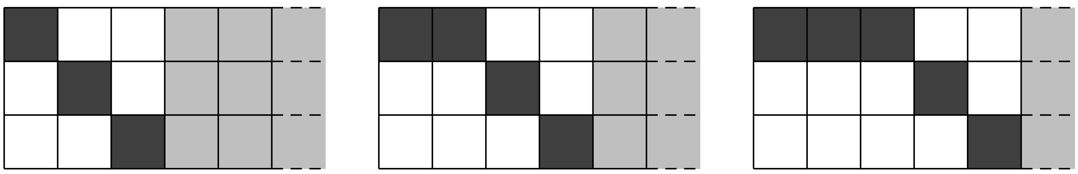


Thus not all patterns of 4 black cells and 12 white cells on a 4×4 grid can be achieved by a covering in which each new tile is placed on top of all previous tiles that it overlaps.

Comment. This pattern can be achieved however if new tiles may be placed under previous tiles.

(iii) By the same argument as that in Part (b) above, the number of black cells exposed in any covering of the $n \times m$ grid is at least m.

We now show m is achievable. Number the columns 1 to m. Using the procedure in Part (a) Method 1 above, cover columns 1 to n to give n black cells on the main diagonal and white cells everywhere else. Now apply the same covering on columns 2 to $n+1$, then on columns 3 to $n+2$, and so on, finishing with columns $m-n+1$ to m. This procedure covers the entire $n \times m$ grid leaving exactly m black cells visible. The following diagram shows this procedure for $n = 3$.



So the minimum number of visible black cells in any covering of the $n \times m$ grid is m. From symmetry, the minimum number of visible white cells in any covering of the $n \times m$ grid is m. Hence the maximum number of visible black cells in any covering of the $n \times m$ grid is $nm - m = m(n-1)$.

AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD STATISTICS

DISTRIBUTION OF AWARDS/SCHOOL YEAR

YEAR	NUMBER OF STUDENTS	NUMBER OF AWARDS				
		Prize	High Distinction	Distinction	Credit	Participation
8	336	2	13	35	97	189
9	390	5	32	62	106	185
10	413	14	42	70	124	163
Other	167	1	9	12	39	106
Total	1306	22	96	179	366	643

NUMBER OF CORRECT ANSWERS QUESTIONS 1–8

YEAR	NUMBER CORRECT/QUESTION							
	1	2	3	4	5	6	7	8
8	132	165	225	157	84	24	10	87
9	173	231	270	221	119	68	21	122
10	209	268	277	226	149	87	23	147
Other	64	79	115	72	37	12	2	38
Total	578	743	887	676	389	191	56	394

MEAN SCORE/QUESTION/SCHOOL YEAR

YEAR	NUMBER OF STUDENTS	MEAN SCORE			OVERALL MEAN
		Question			
		1–8	9	10	
8	336	8.2	0.1	0.1	8.5
9	390	9.8	0.5	0.2	10.5
10	413	10.7	0.6	0.3	11.6
Other	167	7.7	0.2	0.1	7.9
All Years	1306	9.4	0.4	0.2	10.0