AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD

Time allowed: 4 hours. NO calculators are to be used.

Questions 1 to 8 only require their numerical answers all of which are non-negative integers less than 1000.
Questions 9 and 10 require written solutions which may include proofs.
The bonus marks for the Investigation in Question 10 may be used to determine prize winners.

1. Let x denote a single digit. The tens digit in the product of $2x7$ and 39 is 9. Find x. [**2 marks**]

2. If $234_{b+1} - 234_{b-1} = 70_{10}$, what is 234_b in base 10? [**3 marks**]

3. The circumcircle of a square $ABCD$ has radius 10. A semicircle is drawn on AB outside the square. Find the area of the region inside the semicircle but outside the circumcircle. [**3 marks**]

4. Find the last non-zero digit of $50! = 1 \times 2 \times 3 \times \cdots \times 50$. [**3 marks**]

5. Each edge of a cube is marked with its trisection points. Each vertex v of the cube is cut off by a plane that passes through the three trisection points closest to v. The resulting polyhedron has 24 vertices. How many diagonals joining pairs of these vertices lie entirely inside the polyhedron? [**3 marks**]

6. Let $ABCD$ be a parallelogram. Point P is on AB produced such that DP bisects BC at N. Point Q is on BA produced such that CQ bisects AD at M. Lines DP and CQ meet at O. If the area of parallelogram $ABCD$ is 192, find the area of triangle POQ. [**4 marks**]

7. Two different positive integers a and b satisfy the equation $a^2 - b^2 = 2018 - 2a$.
What is the value of $a + b$? [**4 marks**]

8. The area of triangle ABC is 300. In triangle ABC, Q is the midpoint of BC, P is a point on AC between C and A such that $CP = 3PA$, R is a point on side AB such that the area of $\triangle PQR$ is twice the area of $\triangle RBQ$. Find the area of $\triangle PQR$. [**4 marks**]

9. Prove that 38 is the largest even integer that is *not* the sum of two positive odd composite numbers. [**4 marks**]

10. A pair of positive integers is called *compatible* if one of the numbers equals the sum of all digits in the pair and the other number equals the product of all digits in the pair. Find all pairs of positive compatible numbers less than 100. [**5 marks**]

Investigation

Find all pairs of positive compatible numbers less than 1000 with at least one number greater than 99. [**3 bonus marks**]