2019 AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD

Time allowed: 4 hours. NO calculators are to be used.

Questions 1 to 8 only require their numerical answers all of which are non-negative integers less than 1000.
Questions 9 and 10 require written solutions which may include proofs.
The bonus marks for the Investigation in Question 10 may be used to determine prize winners.

1. Gaston and Jordon are two budding chefs who unfortunately always misread the cooking time required for a recipe. For example, if the required cooking time is written as 1:32, meaning 1 hour and 32 minutes, Jordon reads it as 132 minutes while Gaston reads it as 1.32 hours. For one particular recipe, the difference between Jordon's and Gaston's misread times is exactly 90 minutes. What is the actual cooking time in minutes? [**2 marks**]

2. An integer has 3 digits in base 10. When it is interpreted in base 6 and multiplied by 4, the result is 2 more than 3 times the same number when it is interpreted in base 7. What is the largest integer, in base 10, that has this property? [**2 marks**]

3. Let $ABCD$ be a square with side length 24. Let P be the point on side AB with $AP = 8$, and let AC and DP intersect at Q. Determine the area of triangle CQD. [**3 marks**]

4. Jenny has a large bucket of 1000 marbles, of equal size. At least one is red and the rest are green. She calculates that if she picks out two marbles at random, the probability that both are red is the same as the probability that they are different colours. How many red marbles are in the bucket? [**3 marks**]

5. A beetle starts at corner A of a square $ABCD$. Each minute, the beetle moves along exactly one side of the square from one corner of the square to another corner. For example, in the first two minutes, the beetle could have one of 4 walks: ABA, ADA, ABC, ADC. How many walks can the beetle have from corner A to corner C in 10 minutes? [**3 marks**]

6. A leaky boat already has some water on board and water is coming in at a constant rate. Several people are available to operate the manual pumps and, when they do, they all pump at the same rate. Starting at a given time, five people would take 10 hours to pump the boat dry, while 12 people would take only 3 hours. How many people are required to pump it dry in 2 hours from the given time? [**4 marks**]

7. A triangle PQR is to be constructed so that the perpendicular bisector of PQ cuts the side QR at N and the line PN splits the angle QPR into two angles, not necessarily of integer degrees, in the ratio $1:22$. If $\angle QPR = p$ degrees, where p is an integer, find the maximum value of p. [**4 marks**]

8. Find the largest number which will divide $a^5 + b^5 - ab(a^3 + b^3)$ for *all* primes a and b greater than 5. [**4 marks**]

9. In a large botanic garden, there is a long line of 999 regularly spaced positions where plants may grow. At any time, there is at most one plant in each position. In the first year, only the middle position has a plant. In the nth year, for all $n \geq 2$, there is a plant at position X if and only if, in the previous year, the total number of plants in position X and either side of X is precisely one. Determine, with proof, the total number of plants in the 128th year. [**5 marks**]

10. An even number of teams play in a round robin tournament defined as follows. Each team plays every other team exactly once. The matches are arranged in rounds. Each team plays exactly once in each round. We assume there are no draws.

(a) Prove that at the end of round 2, the number of teams that have won two matches equals the number of teams that have lost two matches.

(b) Find the least number of teams such that at the end of round 3, the number of teams that have won three matches may not equal the number of teams that have lost three matches.

(c) Prove that, at the end of round 4, the number of teams that have won 4 matches equals the number of teams that have lost 4 matches, if and only if the number of teams that have won exactly 3 matches equals the number of teams that have lost exactly 3 matches. [**5 marks**]

Investigation

Prove that for all even $k \geq 4$, there is a round robin tournament of k teams in which the number of teams that have won $k-1$ matches does not equal the number of teams that have lost $k-1$ matches. [**3 bonus marks**]

2019 AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD SOLUTIONS

1. *Method 1*

Let the cooking time be $h : m$, where h is the number of hours and m is the number of minutes. Jordon's time is $100h + m$ minutes and Gaston's time is $h + (m/100)$ hours, or $60h + 3m/5$ minutes. Hence the difference of 90 minutes leads to the equation $40h + (2m/5) = 90$, hence $100h + m = 225$. So $0 \le 225 - 100h \le 60$, which implies $h = 2$.

Therefore the actual cooking time in minutes is $2 \times 60 + 25 = \mathbf{145}$.

Method 2

Let m be the number of minutes as read by Jordon. Then Gaston reads this as $m/100$ hours $= 60m/100$ minutes. Hence the difference of 90 minutes leads to the equation $90 = m - 3m/5 = 2m/5$, which gives $m = 225$.

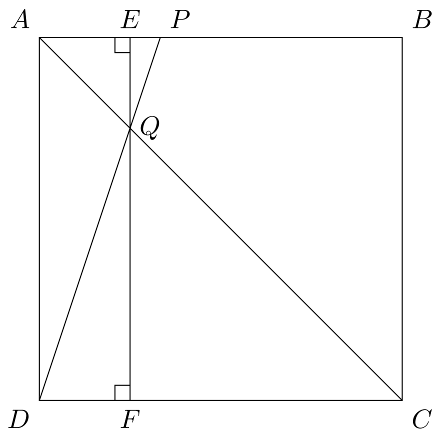
Therefore the actual cooking time in minutes is $2 \times 60 + 25 = \mathbf{145}$.

2. Let the three-digit number be abc. Since $4 \times abc_6 = 3 \times abc_7 + 2$, we have

$$\begin{aligned} 4(36a + 6b + c) &= 3(49a + 7b + c) + 2 \\ 144a + 24b + 4c &= 147a + 21b + 3c + 2 \\ 3b + c &= 3a + 2 \end{aligned}$$

Hence 3 divides $c - 2$. Since $c < 6$, $c = 2$ or 5.
If $c = 2$, then $a = b$. If $c = 5$, then $a = b + 1$.
Since all digits are at most 5, the largest required number (in base 10) is **552**.

3. Draw a line through Q parallel to AD, and let E and F be the intersections of this line with AB and CD, respectively.



Method 1

From similar triangles AEQ and CFQ, we have $EQ/FQ = AQ/CQ$.
From similar triangles APQ and CDQ, we have $AQ/CQ = AP/CD = 8/24 = 1/3$.

So $FQ = \frac{3}{4}FE = \frac{3}{4} \times 24 = 18$.
Hence the area of triangle CQD equals $\frac{1}{2} \times 24 \times 18 = \mathbf{216}$.

Method 2

Let DC and DA be the positive horizontal and vertical cartesian axes with D as the origin.
The equation for line DP is $y = \frac{24}{8}x = 3x$.
The equation for line AC is $y + x = 24$.

So, at Q, $4x = 24$, $x = 4$, and $y = 24 - 6 = 18$.
Hence the area of triangle CQD equals $\frac{1}{2} \times 24 \times 18 = \mathbf{216}$.

4. Let r be the number of red marbles.
The probability of selecting two red marbles is $\dfrac{r}{1000} \times \dfrac{r-1}{999}$.
There are $1000 - r$ green marbles.
So the probability of selecting two marbles of different colour is

$$\frac{r}{1000} \times \frac{1000-r}{999} + \frac{1000-r}{1000} \times \frac{r}{999} = \frac{2r(1000-r)}{1000 \times 999}.$$

Equating the two probabilities gives $r - 1 = 2(1000 - r)$. Hence $3r = 2001$ and $r = \mathbf{667}$.

5. *Method 1*

Each minute, the beetle has the choice of two sides to proceed along. After an odd number of minutes the beetle arrives at B or D.

Therefore, in the first 9 minutes, the beetle could have exactly 2^9 walks, each of which arrives at B or D.

For each such walk, there is then precisely one way to complete it in the tenth minute to arrive at C. So the number of walks from A to C in 10 minutes is $2^9 = \mathbf{512}$.

Method 2

Each minute, the beetle has the choice of two sides to proceed along. After one minute the beetle will arrive at B or D. Starting at B, the beetle will arrive at A or C after an odd number of minutes.

Therefore, in 9 minutes, the beetle could have exactly 2^9 walks from B, each of which arrives at A or C. By symmetry, exactly half of these walks will arrive at C.

So, in 10 minutes, there are exactly 2^8 walks from A to C via B at the end of the first minute. Similarly, there are exactly 2^8 walks from A to C via D at the end of the first minute. So the number of walks from A to C in 10 minutes is $2 \times 2^8 = \mathbf{512}$.

6. Let x be the amount of water in the boat when pumping begins, y the amount of water leaking into the boat per hour, and z the amount each person can pump out of the boat in an hour. Let h be the number of hours needed by n people to pump the boat dry.

Then

$$hy + x = nhz \tag{1}$$

We are given

$$x + 3y = 36z \tag{2}$$
$$x + 10y = 50z \tag{3}$$

Subtracting (2) from (3) gives

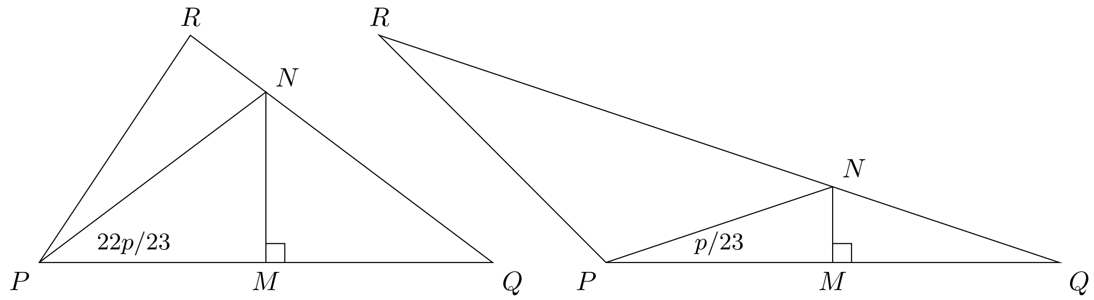
$$\begin{aligned} y &= 2z \\ x &= 30z \end{aligned}$$

So equation (1) becomes

$$\begin{aligned} 30 + 2h &= nh \\ h(n-2) &= 30 \end{aligned}$$

Substituting $h = 2$ gives $n = \mathbf{17}$.

7. Let M be the midpoint of PQ.



Triangles PMN and QMN are congruent (side-angle-side).
Hence $\angle MQN = \angle MPN = 22p/23$ or $p/23$.

The sum of angles in a triangle is 180° and the angle at R must be strictly positive.

If $\angle MPN = 22p/23$, then $p + \frac{22p}{23} < 180$, that is, $p < \frac{23}{45}(180) = 92$.

If $\angle MPN = p/23$, then $p + \frac{p}{23} < 180$, that is, $p < \frac{23}{24}(180) = 172.5$.

If $p = 172$ and $\angle MPN = p/23$, which is less than 7.5, then $\angle PRQ > 0$ and $\triangle PQR$ exists.
So the maximum integer value of p is **172**.

8. Expanding, rearranging, and factorising gives:

$$a^5 + b^5 - a^4b - b^4a = a(a^4 - b^4) - b(a^4 - b^4) = (a-b)(a^4 - b^4) = (a-b)^2(a^2+b^2)(a+b)$$

Let b be any prime greater than 5. Choose a to be some prime greater than 5 but not equal to b. Then b does not divide a. Hence b divides neither $a - b$ nor $a^4 - b^4$. Therefore b cannot divide $(a-b)(a^4-b^4)$. So, if a prime p divides $(a-b)(a^4-b^4)$ for all primes a and b greater than 5, then $p \neq b$. So p is 2, 3, or 5.

As a and b are both odd, both $a+b$ and a^2+b^2 are even and 4 divides $(a-b)^2$. Hence 16 divides the given expression, but we can do better than this. Each of a and b has the form $4m+1$ or $4m-1$. If they have the same form, then 16 divides $(a-b)^2$. If they have different forms, then 4 divides $a+b$. Either way, 32 divides the given expression.

Since 3 divides neither a nor b, each has the form $3m+1$ or $3m-1$. If they have the same form, then 3 divides $a-b$. If they have different forms, then 3 divides $a+b$.

Since 5 divides neither a nor b, each has the form $5m \pm 1$ or $5m \pm 2$. If both have the form $5m \pm 1$ or both have the form $5m \pm 2$, then 5 divides $a^2 - b^2$. If one has the form $5m \pm 1$ and the other has the form $5m \pm 2$, then 5 divides $a^2 + b^2$. Hence 5 divides $a^4 - b^4$.

So the largest guaranteed divisor of the given expression is at least $32 \times 3 \times 5 = 480$.

Substituting $b = 7$ and $a = 13$ in the last factorisation gives $(36)(218)(20) = 2^5 \times 3^2 \times 5 \times 109$. Hence neither 2^6 nor 5^2 is a guaranteed factor of the given expression.

Substituting $b = 11$ and $a = 13$ in the last factorisation gives $(4)(290)(24) = 2^6 \times 3 \times 5 \times 29$. Hence 3^2 is not a guaranteed factor of the given expression.

So the largest guaranteed divisor is **480**.

9. With 1 representing a plant and 0 representing a vacant position, the following list of binary sequences illustrate the plant patterns in the first 9 years. Vacant positions beyond the last plant in either direction are ignored.

Year	Plant Pattern
1	1
2	1 1 1
3	1 0 0 0 1
4	1 1 1 0 1 1 1
5	1 0 0 0 0 0 0 0 1
6	1 1 1 0 0 0 0 0 1 1 1
7	1 0 0 0 1 0 0 0 1 0 0 0 1
8	1 1 1 0 1 1 1 0 1 1 1 0 1 1 1
9	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1

Define the *length* of a binary sequence to be the number of digits it contains.

For $n = 1, 2, 3$, we make two observations.
(1) In year 2^n the plant pattern is $11101110\ldots1110111\ldots01110111$ and has length $2^{n+1} - 1$.
(2) In year $2^n + 1$ the plant pattern is $100\ldots0\ldots001$ and has length $2^{n+1} + 1$.

We use induction to prove these statements are true for *all* $n \geq 1$.

First assume statements (1) and (2) are true for n equal to some positive integer k.

In year 2^k, the length of the plant pattern either side of the centre 0 is $2^k - 1$. In year $2^k + 1$ there are precisely $2^k - 1$ vacant positions between each plant and the centre vacant position. So by assumption, in the year $2^k + 2^k = 2^{k+1}$ the plant pattern will be two copies of the plant pattern for year 2^k separated by precisely one 0. Thus in year 2^{k+1} the plant pattern will be $1110\ldots1110111\ldots0111$ and have length $2(2^{k+1} - 1) + 1 = 2^{k+2} - 1$. Then, in year $2^{k+1} + 1$, the plant pattern will be $100\ldots0\ldots001$ and have length $2^{k+1} + 1$.

Thus statements (1) and (2) are true for $n = k + 1$, and therefore true for all $n \geq 1$.

Since $128 = 2^7$, the plant pattern for year 128 is $11101110\ldots1110111\ldots01110111$ and has length $2^8 - 1$. If we place a 0 at the right-hand end of this sequence, we will have 2^8 digits of which exactly $2^8/4$ will be 0. So the number of plants in year 128 will be $2^8 - 2^8/4 = 2^8 - 2^6 = 2^6(4 - 1) = \mathbf{192}$.

10. (a)
Method 1

After two rounds, each team has two wins, or two losses, or one win and one loss. Since the total number of wins equals the total number of losses and the last category of teams accounts for an equal number of wins and losses, the number of teams with two wins must equal the number of teams with two losses.

Method 2

At the end of round 2, let r_i equal the number of teams that have won i matches. Then r_1 also equals the number of teams that have lost one match and r_0 equals the number of teams that have lost two matches. Since the total number of wins equals the total number of losses, we have $2r_2 + r_1 = r_1 + 2r_0$. Hence $r_2 = r_0$.

Method 3

If a team scores 1 if it wins and -1 if it loses, then the total of all team scores after each round is zero. Hence the total of all team scores after any number of rounds is zero. After two rounds a team may have -2, 0 or 2 points. Hence the number of teams with 2 points must equal the number of teams with -2 points.

Comment

The statement in part (a) is not necessarily true if drawn matches are allowed. For example, with 4 teams, one team may win two matches and the other 2 matches may be drawn.

(b) If there are three rounds, then there are at least 4 teams. In the following rounds there are 4 teams denoted 1, 2, 3, 4, and $a \to b$ indicates team a defeated team b.

$$\begin{array}{lll} \text{Round 1:} & 1 \to 2, & 3 \to 4 \\ \text{Round 2:} & 1 \to 3, & 4 \to 2 \\ \text{Round 3:} & 1 \to 4, & 2 \to 3 \end{array}$$

Thus one team wins 3 matches and no team loses 3 matches. So the least number of teams such that at the end of round 3 the number of teams that have won three matches does not necessarily equal the number of teams that have lost three matches is **4**.

(c) At the end of round 4, let r_i equal the number of teams that have won i matches and let s_i equal the number of teams that have lost i matches. Since the total number of wins equals the total number of losses, we have $4r_4 + 3r_3 + 2r_2 + r_1 = 4s_4 + 3s_3 + 2s_2 + s_1$.

Since a team wins i matches if and only if it loses $4 - i$ matches, we have $r_1 = s_3$, $r_2 = s_2$, $r_3 = s_1$, and $4r_4 + 3r_3 + 2r_2 + s_3 = 4s_4 + 3s_3 + 2r_2 + r_3$. Hence $4r_4 + 2r_3 = 4s_4 + 2s_3$. So $r_4 = s_4$ if and only if $r_3 = s_3$.

Investigation

Method 1

For k teams, we generalise the round in part (b) above in which team 1 defeats all other teams and every team has at least one win. In each column except the first, the winning and losing teams cycle through the sequence 2, 3, 4, $\ldots$, k. For a tidier table, we let $k = 2n$.

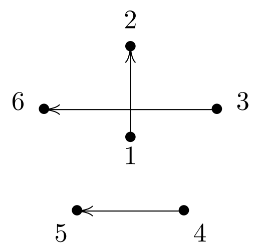
Rd 1:	$1 \to 2$,	$3 \to 2n$,	$4 \to 2n-1$,	$\dots,$	$n+1 \to n+2$
Rd 2:	$1 \to 3$,	$4 \to 2$,	$5 \to 2n$,	$\dots,$	$n+2 \to n+3$
Rd 3:	$1 \to 4$,	$5 \to 3$,	$6 \to 2$,	$\dots,$	$n+3 \to n+4$
$\vdots$					
Rd $n-1$:	$1 \to n$,	$n+1 \to n-1$,	$n+2 \to n-2$,	$\dots,$	$2n-1 \to 2n$
Rd n:	$1 \to n+1$,	$n+2 \to n$,	$n+3 \to n-1$,	$\dots,$	$2n \to 2$
Rd $n+1$:	$1 \to n+2$,	$n+3 \to n+1$,	$n+4 \to n$,	$\dots,$	$2 \to 3$
$\vdots$					
Rd $2n-3$:	$1 \to 2n-2$,	$2n-1 \to 2n-3$,	$2n \to 2n-4$,	$\dots,$	$n-2 \to n-1$
Rd $2n-2$:	$1 \to 2n-1$,	$2n \to 2n-2$,	$2 \to 2n-3$,	$\dots,$	$n-1 \to n$
Rd $2n-1$:	$1 \to 2n$,	$2 \to 2n-1$,	$3 \to 2n-2$,	$\dots,$	$n \to n+1$

It is straightforward to check that each team plays exactly once in each round and each team plays every other team exactly once in the tournament. From column 1, team 1 wins all its matches. From column 2, each of teams 2 to k win at least one match. Thus one team wins $k-1$ matches and no team loses $k-1$ matches.

Method 2

Denote the teams by 1, 2, 3, $\dots$, k. Represent each team by a unique appropriately labelled vertex in a graph. Draw an arrow from a to b to indicate that team a defeated team b.

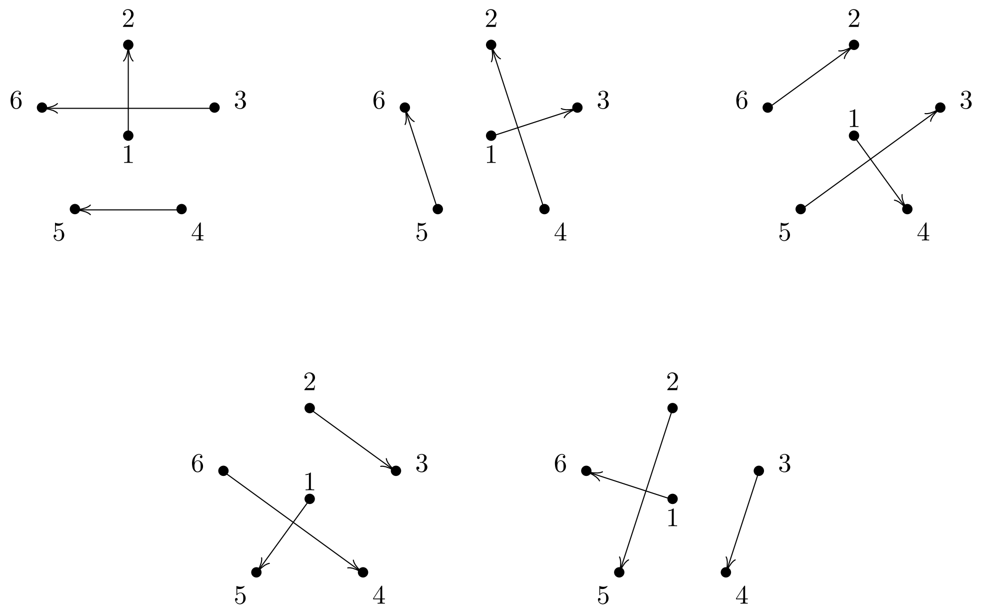
Place vertices 2, 3, $\dots$, k clockwise evenly spaced in a circle with vertex 1 at its centre. Draw an arrow from 1 to i, then $i+1$ to $i-1$, $i+2$ to $i-2$, and so on until all vertices are exhausted. Note that the first arrow is perpendicular to all the other arrows. This represents a legitimate round in the tournament, as shown here for $k = 6$ and $i = 2$.



Now rotate the set of arrows so there is an arrow from 1 to j. This represents another legitimate round that has no match in common with the previous round.

Rotating the set of arrows so that $i = 2, 3, \dots, k$ gives all $k-1$ rounds of a legitimate tournament. In this tournament, team 1 wins $k-1$ matches and each team wins at least one match so none loses $k-1$ matches.

The procedure for $k = 6$ is illustrated here:



2019 AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD STATISTICS

Distribution of Awards/School Year

Year	Number of Students	Number of Awards				
		Prize	High Distinction	Distinction	Credit	Participation
8	466	12	18	48	102	286
9	515	18	31	83	175	208
10	532	26	31	124	194	157
Other	356	3	7	24	72	250
All Years	**1869**	**59**	**87**	**279**	**543**	**901**

The award distribution is based on approximately the top 10% for High Distinction, next 15% for Distinction and the following 25% for Credit.

Number of Correct Answers Questions 1–8

Year	Number Correct / Question							
	1	2	3	4	5	6	7	8
8	331	41	257	144	176	143	31	14
9	408	63	332	257	275	224	56	28
10	440	94	355	320	314	266	69	29
Other	236	20	150	82	112	85	11	5
All Years	**1415**	**218**	**1094**	**803**	**877**	**718**	**167**	**76**

Mean Score/Question/School Year

School Year	Number of Students	Question			Overall Mean
		1–8	9	10	
8	466	8.6	0.5	0.8	9.3
9	515	11.1	0.8	1.0	12.5
10	532	12.4	1.0	1.1	14.2
Other	356	6.5	0.3	0.7	7.3
All Years	**1869**	**10.0**	**0.7**	**0.9**	**11.2**