AUSTRALIAN MATHS TRUST

2020 Australian Intermediate Mathematics Olympiad

Questions

1. If n is a positive integer and n^2 equals the 4-digit number $\underline{aabb}$, find n.

[**2 marks**]

2. Two operations L and R are defined as follows on rational numbers $\frac{p}{q}$, where p and q are positive integers:

$$L\left(\frac{p}{q}\right) = \frac{p}{p+q} \qquad \text{and} \qquad R\left(\frac{p}{q}\right) = \frac{p+q}{q}.$$

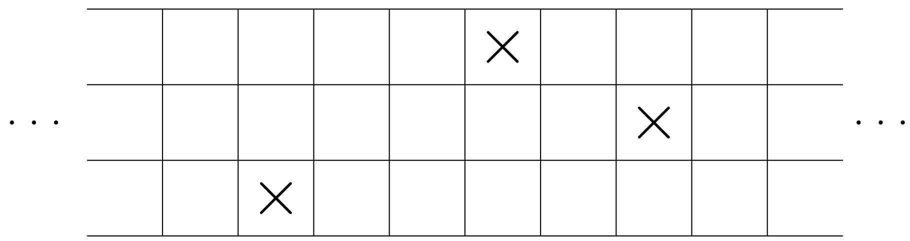
Start from 1 and apply the operations $R, L, R, L, R, L, R, L, R, L$ successively. When the result is written as a fraction in simplest form, what is the sum of its numerator and denominator?

[**2 marks**]

3. Three friends in year 9, Jan, Kate and Lee, sit in that order in the same row at assembly. Each row in the assembly hall has 30 seats numbered 1 to 30.

There are $3n$ year 9 lockers. They are arranged in three rows and numbered left to right from 1 to n in the top row, $n+1$ to $2n$ in the middle row, and $2n+1$ to $3n$ in the bottom row.

The three friends' lockers are located like this:



The girls notice that Kate's assembly seat number divides each of their locker numbers. What is Kate's seat number?

[**3 marks**]

4. $ABCD$ is a square of side 10 cm. E, F, G, H are points on the sides AB, BC, CD, DA respectively. Given that $EB = FC$, $CG = DH$, and $CG - EB = 4$ cm, find the area of the quadrilateral $EFGH$ in square centimetres.

[**3 marks**]

5. Find the largest 3-digit number $N = \underline{abc}$ such that for an integer $d \geq 0$ with $d \leq b$ and $d \leq c$, if a is increased by d and b and c are decreased by d, then the result is a number equal to $Nd/2$.

[**3 marks**]

PLEASE TURN OVER THE PAGE FOR QUESTIONS 6, 7, 8, 9, AND 10

AUSTRALIAN MATHS TRUST

6. Find the value of a in the solution of the following system of equations:

$$a + b + c = 2020 \tag{1}$$
$$a^2 + ac = b^2 + bc \tag{2}$$
$$a^2 + ab = c^2 + cb - 2020 \tag{3}$$

[**4 marks**]

7. A circle with centre C and radius 36 and a circle with centre D and radius 9 touch externally. They lie above a common horizontal tangent which meets the first circle at A and the second circle at B. A circle with centre E is tangent to these two circles and to the segment AB. Find the area of triangle CDE.

[**4 marks**]

8. Proceeding through a sequence of numbers term by term, we calculate a running tally as follows. The tally starts at zero. Starting with the first term, a term is subtracted from the running tally if the result is non-negative, otherwise it is added to the tally. When we arrive at the end of the sequence, the resulting tally is called the *roman sum* of the sequence. For instance, the roman sum of the sequence 2, 4, 3, 3, 1, 5 is $0 + 2 + 4 - 3 - 3 + 1 + 5 = 6$.

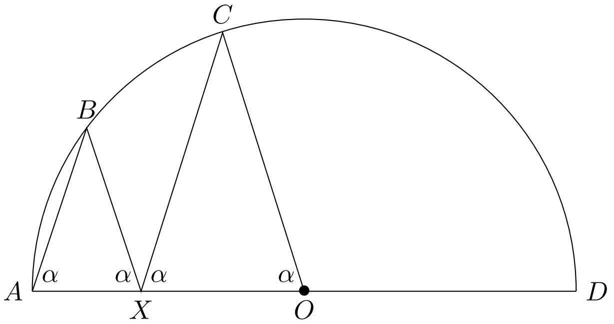
For a sequence consisting of the numbers 1, 2, 3, ... , 100 in some order, what is its largest possible roman sum?

[**4 marks**]

9. If k, m, n are integers such that $2 \le k \le m < n$, show that $\dfrac{m!}{m^k(m-k)!} < \dfrac{n!}{n^k(n-k)!}$.
Note that $0! = 1$ and $r! = 1 \times 2 \times 3 \times \cdots \times r$ for any positive integer r.

[**5 marks**]

10. A circle with centre O has diameter AD. With X on AO and points B and C on the circle, triangles ABX and XCO are similar isosceles with base angles α as shown. Find, with proof, the value of α.



[**5 marks**]

Investigation

(a) Find α if, instead of two similar isosceles triangles on AO, there are three.

[**1 bonus mark**]

(b) Find α if there are n similar isosceles triangles along AO.

[**2 bonus marks**]