AUSTRALIAN MATHS TRUST

2020 Australian Intermediate Mathematics Olympiad

Questions

1. If n is a positive integer and n^2 equals the 4-digit number $\underline{aabb}$, find n.

[**2 marks**]

2. Two operations L and R are defined as follows on rational numbers $\frac{p}{q}$, where p and q are positive integers:

$$L\left(\frac{p}{q}\right) = \frac{p}{p+q} \qquad \text{and} \qquad R\left(\frac{p}{q}\right) = \frac{p+q}{q}.$$

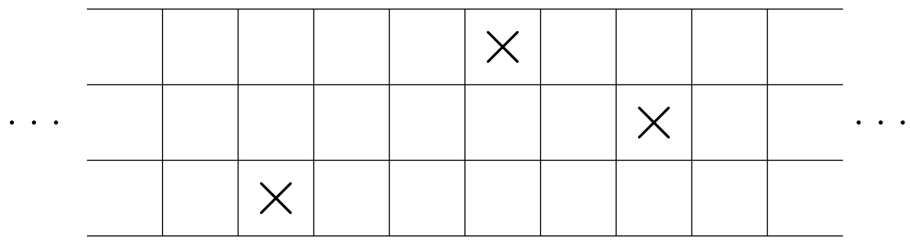
Start from 1 and apply the operations $R, L, R, L, R, L, R, L, R, L$ successively. When the result is written as a fraction in simplest form, what is the sum of its numerator and denominator?

[**2 marks**]

3. Three friends in year 9, Jan, Kate and Lee, sit in that order in the same row at assembly. Each row in the assembly hall has 30 seats numbered 1 to 30.

There are $3n$ year 9 lockers. They are arranged in three rows and numbered left to right from 1 to n in the top row, $n+1$ to $2n$ in the middle row, and $2n+1$ to $3n$ in the bottom row.

The three friends' lockers are located like this:



The girls notice that Kate's assembly seat number divides each of their locker numbers. What is Kate's seat number?

[**3 marks**]

4. $ABCD$ is a square of side 10 cm. E, F, G, H are points on the sides AB, BC, CD, DA respectively. Given that $EB = FC$, $CG = DH$, and $CG - EB = 4$ cm, find the area of the quadrilateral $EFGH$ in square centimetres.

[**3 marks**]

5. Find the largest 3-digit number $N = \underline{abc}$ such that for an integer $d \geq 0$ with $d \leq b$ and $d \leq c$, if a is increased by d and b and c are decreased by d, then the result is a number equal to $Nd/2$.

[**3 marks**]

6. Find the value of a in the solution of the following system of equations:

$$a + b + c = 2020 \quad (1)$$
$$a^2 + ac = b^2 + bc \quad (2)$$
$$a^2 + ab = c^2 + cb - 2020 \quad (3)$$

[**4 marks**]

7. A circle with centre C and radius 36 and a circle with centre D and radius 9 touch externally. They lie above a common horizontal tangent which meets the first circle at A and the second circle at B. A circle with centre E is tangent to these two circles and to the segment AB. Find the area of triangle CDE.

[**4 marks**]

8. Proceeding through a sequence of numbers term by term, we calculate a running tally as follows. The tally starts at zero. Starting with the first term, a term is subtracted from the running tally if the result is non-negative, otherwise it is added to the tally. When we arrive at the end of the sequence, the resulting tally is called the *roman sum* of the sequence. For instance, the roman sum of the sequence 2, 4, 3, 3, 1, 5 is $0 + 2 + 4 - 3 - 3 + 1 + 5 = 6$.

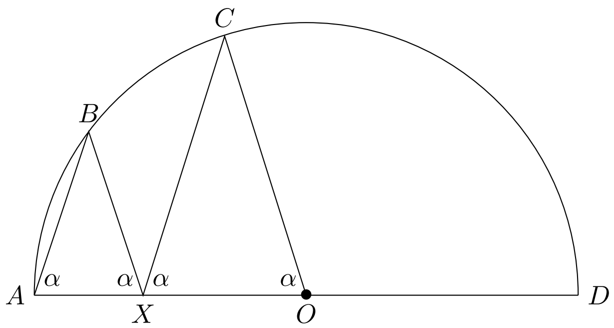
For a sequence consisting of the numbers 1, 2, 3, ... , 100 in some order, what is its largest possible roman sum?

[**4 marks**]

9. If k, m, n are integers such that $2 \le k \le m < n$, show that $\dfrac{m!}{m^k(m-k)!} < \dfrac{n!}{n^k(n-k)!}$.
Note that $0! = 1$ and $r! = 1 \times 2 \times 3 \times \cdots \times r$ for any positive integer r.

[**5 marks**]

10. A circle with centre O has diameter AD. With X on AO and points B and C on the circle, triangles ABX and XCO are similar isosceles with base angles α as shown. Find, with proof, the value of α.



[**5 marks**]

Investigation

(a) Find α if, instead of two similar isosceles triangles on AO, there are three.

[**1 bonus mark**]

(b) Find α if there are n similar isosceles triangles along AO.

[**2 bonus marks**]

1. *Method 1*

Since $a - a + b - b = 0$, we know that $\underline{aabb}$ is divisible by 11. Therefore n is divisible by 11.

Since n^2 has 4 digits, $n = 33, 44, 55, 66, 77, 88$, or 99.
Of these, only $88^2 = 7744$ has the form $\underline{aabb}$. So $n = \mathbf{88}$.

Method 2

A square number ends in 0, 1, 4, 5, 6, or 9 and has remainder 0 or 1 when divided by 4.
When divided by 4, $\underline{aa11}$, $\underline{aa55}$, and $\underline{aa99}$ all have remainder 3, and $\underline{aa66}$ has remainder 2.
So n^2 must be of the form $\underline{aa00}$ or $\underline{aa44}$. If $\underline{aa00}$ is a square number, than $\underline{aa}$ must be a square number, which is impossible. So n^2 must be of the form $\underline{aa44}$.

Since $\underline{aa44} = 11 \times \underline{a04}$ and 11 is prime, 11 divides $\underline{a04} = 100a + 4 = 99a + (a + 4)$.
Therefore 11 divides $a + 4$. Hence $a = 7$, $n^2 = 7744$, and $n = \mathbf{88}$.

Method 3

Since $n^2 = \underline{aabb} = \underline{a0b} \times 11$ and 11 is prime, 11 also divides $\underline{a0b}$ and $\underline{a0b}/11$ is a square number.
From the divisibility test for 11, we have $a + b = 11$.

Since square numbers end in 0, 1, 4, 5, 6, or 9 the only possibilities for $\underline{a0b}$ are 209, 506, 605, 704, and these give 19, 46, 55, 64 when divided by 11. The only square among these is 64, so $n^2 = 8^2 \times 11^2 = 7744$ and $n = 8 \times 11 = \mathbf{88}$.

2. *Method 1*

Applying R followed by L to $\frac{p}{q}$, we get $\frac{p+q}{p+2q}$. We now do this 5 times starting with $1 = 1/1$.

$L(R(1/1)) = 2/3$.
$L(R(2/3)) = 5/8$.
$L(R(5/8)) = 13/21$.
$L(R(13/21)) = 34/55$.
$L(R(34/55)) = 89/144$.

Since 89 is prime, the required number is $89 + 144 = \mathbf{233}$.

Method 2

Starting with $1 = 1/1$, we get:

$$\begin{array}{ll}
R(1/1) = 2/1 & L(2/1) = 2/3 \\
R(2/3) = 5/3 & L(5/3) = 5/8 \\
R(5/8) = 13/8 & L(13/8) = 13/21 \\
R(13/21) = 34/21 & L(34/21) = 34/55 \\
R(34/55) = 89/55 & L(89/55) = 89/144
\end{array}$$

Since 89 is prime, the required number is $89 + 144 = \mathbf{233}$.

Comment

Note that $L(R(p/q)) = \frac{p+q}{(p+q)+q}$. So, starting with 1/1 and applying R followed by L successively produces a sequence of fractions whose numerators and denominators give the Fibonacci sequence 1, 1, 2, 3, 5, 8, ...

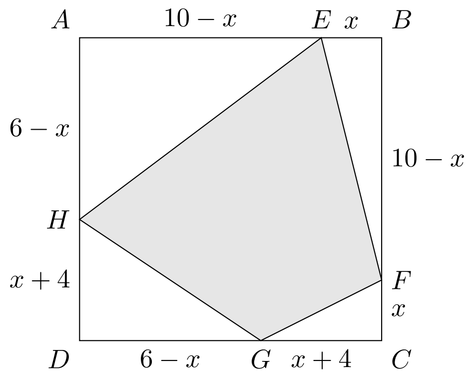
3. Let the locker in the bottom row be x lockers from the left end of the row. Then the three locker numbers are $x+3$, $n+x+5$, and $2n+x$.

Let Kate's assembly seat number be y. Then y divides each of $x+3$, $n+x+5$, and $2n+x$. Hence y divides the differences $(n+x+5)-(x+3)=n+2$, $(2n+x)-(n+x+5)=n-5$, and $(n+2)-(n-5)=7$.

So y is either 1 or 7. Since Kate sits in the middle of the three girls, her assembly seat number is at least 2. Therefore Kate's assembly seat number is **7**.

4. *Method 1*

Let $EB=x$. Then we have the lengths shown (in centimetres).



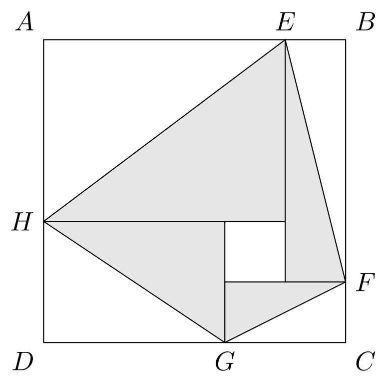
The total area of triangles EBF, FCG, GDH, HAE is

$$(x(10-x)+x(x+4)+(6-x)(x+4)+(6-x)(10-x))/2=(24+60)/2=42.$$

Hence the area of $EFGH$ in square centimetres is $100-42=\mathbf{58}$.

Method 2

Draw horizontal and vertical lines as shown.



The area of the small white rectangle is $(DH-FC)(CG-EB)=(CG-EB)^2=4^2=16\,\text{cm}^2$. Each shaded triangle is congruent to its adjacent white triangle. So double the shaded area is $100-16=84\,\text{cm}^2$.

Hence the area of $EFGH$ in square centimetres is $16+\frac{1}{2}(84)=16+42=\mathbf{58}$.

5. The old number is $N = 100a + 10b + c$ and the new number is
$Nd/2 = 100(a + d) + 10(b - d) + (c - d) = N + 89d$.
So $N(d - 2) = 2 \times 89d$.

Method 1

If $d = 0$ or 1, then the left side is negative but the right side is not.
If $d = 2$, then the left side is 0 but the right side is not.
If $d = 3$, then $N = 6 \times 89 = 534$.
If $d = 4$, then $N = 4 \times 89 = 356$.
If $d = 5$, then the left side is a multiple of 3 but the right side is not.
If $d = 6$, then $N = 3 \times 89 = 267$.
If $d = 7$, then the left side is a multiple of 5 but the right side is not.
If $d = 8$, then the left side is a multiple of 3 but the right side is not.
If $d = 9$, then the left side is a multiple of 7 but the right side is not.

So the largest value of N is **534**.

Method 2

If $d = 0$ or 1, then the left side is negative but the right side is not.
If $d = 2$, then the left side is 0 but the right side is not.
If $d > 2$, then

$$N = 178\left(\frac{d}{d-2}\right) = 178\left(1 + \frac{2}{d-2}\right)$$

which decreases with increasing d.

So the largest value of N is $178\left(1 + \frac{2}{3-2}\right) = \mathbf{534}$.

Method 3

We have

$$\begin{aligned} Nd &= 2N + 178d \\ Nd - 2N - 178d &= 0 \\ (N - 178)(d - 2) &= 356 \end{aligned}$$

Since $d - 2$ is an integer, $N - 178 \leq 356$ and $N \leq 534$.
Since $N = 534$ if $d = 3$, the largest value of N is **534**.

Method 4

Since 89 is prime and $-2 \leq d - 2 \leq 7$, 89 divides N.
Let $N = 89k$. Then $k(d - 2) = 2d$.

If $d = 0$, then $k = 0$.
If $d = 1$, then $k = -2$.
If $d = 2$, then there is no solution for k.
If $d > 2$, then $k = 2d/(d - 2) = 2 + 4/(d - 2)$, which decreases with increasing d.
So the largest value of N is $89(2 + 4/(3 - 2)) = 89(6) = \mathbf{534}$.

6. *Method 1*

Rearranging equation (2), then using equation (1), we have

$$\begin{aligned} a^2 - b^2 &= bc - ac \\ (a+b)(a-b) &= c(b-a) \\ (a-b)(a+b+c) &= 0 \\ (a-b)2020 &= 0 \end{aligned}$$

So $b = a$.

Rearranging equation (3), then using equation (1), we have

$$\begin{aligned} 2020 &= c^2 - a^2 + cb - ab \\ &= (c-a)(a+b+c) \\ &= (c-a)2020 \end{aligned}$$

So $c = a + 1$.

From equation (1), $2020 = a + a + (a+1) = 3a + 1$, hence $a = 2019/3 = \mathbf{673}$.

Method 2

As in Method 1, $b = a$.

Substituting $b = a$ in equation (1) gives $c = 2020 - 2a$. Then equation (3) gives

$$\begin{aligned} 2a^2 &= (2020 - 2a)^2 + (2020 - 2a)a - 2020 \\ 0 &= 2020^2 - 8080a + 2020a - 2020 \\ 0 &= 2020 - 3a - 1 \\ 3a &= 2019 \end{aligned}$$

Hence $a = 2019/3 = \mathbf{673}$.

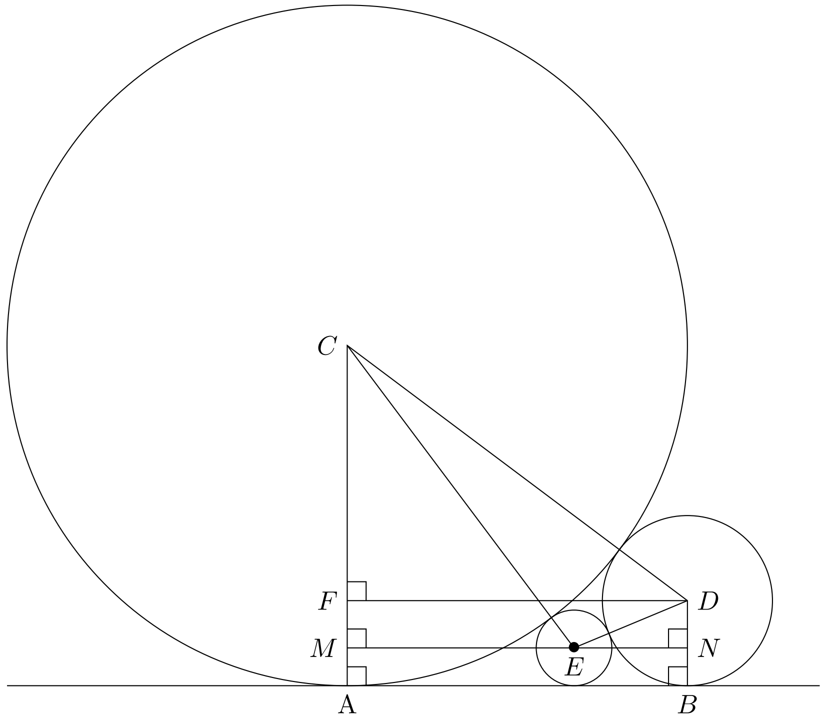
Method 3

Adding equations (2) and (3) then using equation (1), we have

$$\begin{aligned} 2a^2 + ab + ac &= b^2 + c^2 + 2bc - 2020 \\ 2a^2 + a(b+c) &= (b+c)^2 - 2020 \\ 2a^2 + a(2020 - a) &= (2020 - a)^2 - 2020 \\ 2020a &= 2020^2 - 4040a - 2020 \\ a &= 2020 - 2a - 1 \\ 3a &= 2019 \end{aligned}$$

Hence $a = 2019/3 = \mathbf{673}$.

7. Let MEN and DF be lines parallel to AB as shown.



By Pythagoras, $DF^2 = DC^2 - CF^2 = (36+9)^2 - (36-9)^2 = 4 \times 9 \times 36 = 36^2$.

Let r be the radius of the circle with centre E.
Then $CM = 36 - r$, $CE = 36 + r$, and $ME^2 = (36+r)^2 - (36-r)^2 = 144r$.
Also $DE = 9 + r$, $DN = 9 - r$, and $NE^2 = (9+r)^2 - (9-r)^2 = 36r$.

So we have $36 = DF = MN = ME + EN = \sqrt{144r} + \sqrt{36r} = 12\sqrt{r} + 6\sqrt{r} = 18\sqrt{r}$.
Hence $r = (36/18)^2 = 4$.

Method 1

$$\begin{aligned}
|CDE| &= |DFC| + |MNDF| - |CME| - |DNE| \\
&= \frac{1}{2}DF.CF + MN.ND - \frac{1}{2}ME.MC - \frac{1}{2}NE.ND \\
&= \frac{1}{2}.36.27 + 36.5 - \frac{1}{2}.24.32 - \frac{1}{2}.12.5 \\
&= 18.27 + 36.5 - 12.32 - 6.5 \\
&= 6(81 + 30 - 64 - 5) \\
&= 6(42) = \mathbf{252}.
\end{aligned}$$

Method 2

We have $DE = 9 + 4 = 13$, $EC = 4 + 36 = 40$, and $CD = 36 + 9 = 45$.
Let $S = (DE + EC + CD)/2 = 98/2 = 49$.

By Heron's formula,

$$\begin{aligned}
|CDE| &= \sqrt{49(49-45)(49-40)(49-13)} \\
&= \sqrt{49 \times 4 \times 9 \times 36} \\
&= 7 \times 2 \times 3 \times 6 \\
&= \mathbf{252}.
\end{aligned}$$

8. If the running tally T is ever 100 or larger, then the next term (if any) in the sequence will be subtracted from T. So if we add to T (the only other option), then T must be less than 100. Hence the running tally can never get larger than $99 + 100 = 199$. Thus the roman sum of any sequence consisting of 1, 2, 3, ... , 100 in some order is at most 199.

We now show that the roman sum cannot be 199. Since exactly 50 of the numbers 1 to 100 are odd, the roman sum will consist of the sum of 50 odd integers (some possibly negative), which will be even, plus the sum of 50 even integers (some possibly negative), which is also even. As the sum of two even integers is even, the roman sum cannot be 199.

A roman sum of 198 can be achieved by keeping the running tally close to 0 until the last few terms. In the following example, every four terms, except the last four, have a tally of 0. The last four terms then result in a roman sum of 198:

$$(1 + 4 - 2 - 3) + (5 + 8 - 6 - 7) + \cdots + (93 + 96 - 94 - 95) + (97 + 99 - 98 + 100) = 198.$$

So the largest possible roman sum for a sequence consisting of 1, 2, 3, ... , 100 in some order is **198**.

9. *Method 1*

We first note that for any non-negative integer i we have $mi \le ni$ with equality if and only if $i = 0$.

Hence $mn - ni \le mn - mi$. So $(m - i)n \le (n - i)m$, or equivalently

$$\frac{m-i}{m} \le \frac{n-i}{n}$$

with equality if and only if $i = 0$.

We now multiply together such inequalities, for integers i from 0 to $k - 1$. This gives

$$\frac{m}{m}\frac{m-1}{m}\cdots\frac{m-k+1}{m} \le \frac{n}{n}\frac{n-1}{n}\cdots\frac{n-k+1}{n}$$

Since $k \ge 2$, there is at least one factor where $i > 0$, so the above inequality is strict and we have

$$\frac{m(m-1)\cdots(m-k+1)}{m^k} < \frac{n(n-1)\cdots(n-k+1)}{n^k}$$

Hence

$$\frac{m!}{m^k(m-k)!} < \frac{n!}{n^k(n-k)!}$$

Method 2

Let $n = m + r$ where $r > 0$.

Then for $k \geq 2$,

$$\frac{m!}{m^k(m-k)!} < \frac{n!}{n^k(n-k)!}$$

is equivalent to

$$\frac{m!}{m^k(m-k)!} < \frac{(m+r)!}{(m+r)^k(m+r-k)!}$$

which is equivalent to

$$\left(1+\frac{r}{m}\right)^k < \frac{(m+r)(m+r-1)\cdots(m+r-k+1)}{(m)(m-1)\cdots(m-k+1)}$$

The right side equals

$$\left(1+\frac{r}{m}\right)\left(1+\frac{r}{m-1}\right)\ldots\left(1+\frac{r}{m-k+1}\right)$$

Since $1+\dfrac{r}{m}$ is less than each of $1+\dfrac{r}{m-1}, \ldots, 1+\dfrac{r}{m-k+1}$, the last inequality is valid.
Hence the first inequality is valid.

Method 3

To prove $\dfrac{m!}{m^k(m-k)!} < \dfrac{n!}{n^k(n-k)!}$ for $2 \leq k \leq m < n$, it suffices to prove this for $n = m + 1$.

Substituting $n = m + 1$ and simplifying gives the equivalent inequality

$$(m+1)^{k-1}(m+1-k) < m^k.$$

We prove this by induction on k for $2 \leq k \leq m$.

The inequality is true for $k = 2$, since the left side is $(m+1)(m-1) = m^2 - 1 < m^2$.

Now assume the inequality is true for $k = r < m$. That is, $(m+1)^{r-1}(m+1-r) < m^r$. Multiplying both sides by m gives

$$\begin{aligned}
m^{r+1} &> m(m+1)^{r-1}(m+1-r) \\
&= \frac{m}{m+1}(m+1)^r(m-r)\frac{m+1-r}{m-r} \\
&= \frac{m^2+m-mr}{m^2+m-mr-r}(m+1)^r(m-r) \\
&> (m+1)^r(m-r)
\end{aligned}$$

Thus the required inequality is true for $k = r + 1$. By induction it is true for $2 \leq k \leq m$.

Method 4

To prove $\dfrac{m!}{m^k(m-k)!} < \dfrac{n!}{n^k(n-k)!}$ for $2 \le k \le m < n$, it suffices to prove this for $n = m+1$.

Substituting $n = m+1$ and simplifying gives the equivalent inequality

$$(m+1)^{k-1}(m+1-k) < m^k,$$

which is equivalent to

$$\sqrt[k]{(m+1)^{k-1}(m+1-k)} < m.$$

This follows from the inequality of arithmetic and geometric means (GM $\le$ AM).

The left side is the geometric mean of the k numbers $m+1, \ldots, m+1, m+1-k$. Their arithmetic mean is $((k-1)(m+1) + (m+1-k))/k = km/m = m$, which is the right side.

The inequality is strict because $m+1-k \ne m+1$.

Comment

Here are two probability interpretations of the given inequality.

1. Suppose a bowl contains m red sweets and $n-m$ blue sweets. If k sweets are selected from the bowl with replacement, the probability they will all be red is $(m/n)^k$. If the selection is without replacement, the probability is

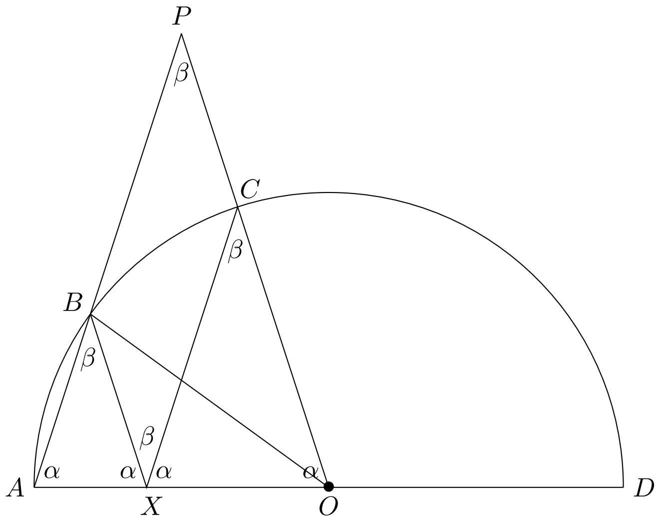
$$\frac{m}{n}\frac{m-1}{n-1}\cdots\frac{m-k+1}{n-k+1} = \frac{m!}{(m-k)!}\frac{(n-k)!}{n!}$$

So the inequality $\dfrac{m!}{m^k(m-k)!} < \dfrac{n!}{n^k(n-k)!}$ tells us that the probability of all red sweets without replacement is less than the probability with replacement, which is intuitively true.

2. Suppose a fair m-sided die is rolled k times. The probability of obtaining a sequence of k distinct numbers is $\dfrac{m!}{m^k(m-k)!}$. The analogous probability with an n-sided die is $\dfrac{n!}{n^k(n-k)!}$. The inequality tells us that if $m < n$, then the first probability is less than the second.

10. *Method 1*

Extend AB and OC to meet at P. Draw OB.
Let $\angle ABX = \beta$. Then $2\alpha + \beta = 180°$.
So each of the angles XCO, APO, BXC equals β.



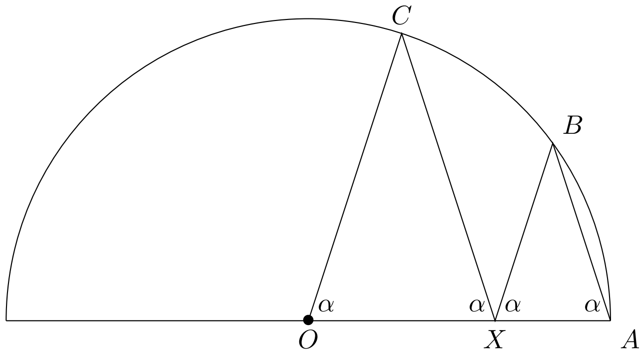
Since $\angle OAB = \angle OXC$, $AP || XC$.
Since $\angle AXB = \angle AOP$, $XB || OP$.
So $BPCX$ is a parallelogram. Hence $BP = XC = OC = OB$.

Therefore $\triangle PBO$ is isosceles and $\angle BOP = \angle BPO = \beta$.
Hence $BCOX$ is cyclic and therefore $\angle XBO = \angle XCO = \beta$.

Since $OA = OB$, $\triangle OAB$ is isosceles and $\alpha = \angle ABC = 2\beta$.
This gives $5\beta = 180°$, hence $\alpha = 72°$.

Method 2

Place the circle on the cartesian plane with its centre at the origin and OA on the positive horizontal axis. Let the radius of the circle be 1.



We have these coordinates: C $(\cos\alpha,\ \sin\alpha)$, X $(2\cos\alpha,\ 0)$, A (1,0).

Let $t = \cos\alpha$, and $k = AX/XO = (1-2t)/2t$.
Then the coordinates of B are $(2t + kt,\ k\sin\alpha)$.

Since B is on the circle, we have

$$\begin{aligned}
1 &= (2t+kt)^2 + (k\sin\alpha)^2 \\
&= (2t+kt)^2 + k^2(1-t^2) \\
&= 4t^2 + 4kt^2 + k^2 \\
&= 4t^2 + 2t(1-2t) + (1-2t)^2/4t^2 \\
4t^2 &= 8t^3 + (1-2t)^2 \\
0 &= 8t^3 - 4t + 1
\end{aligned}$$

The last equation has three solutions for t: $1/2$, $(-1 \pm \sqrt{5})/4$.
Of these only $(\sqrt{5}-1)/4$ fits the geometry.
From trigonometry, $\cos 72^\circ = (\sqrt{5}-1)/4$. Hence $\alpha = 72^\circ$.

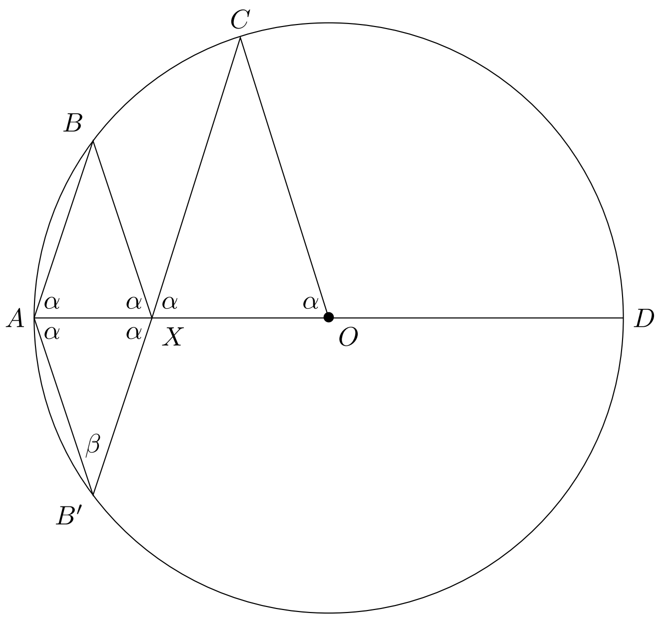
Comment

It is acknowledged that this trigonometric solution is probably inaccessible to most year 10 students.

Method 3

Reflect triangle ABX in diameter AD to give triangle $AB'X$.
Note that $B'XC$ is a straight line.

Let $\angle AB'C = \beta$.
Note that $2\alpha + \beta = 180°$.



Since the angle subtended by an arc of a circle at its centre is twice the angle subtended by the arc at the circumference, we have $\alpha = \angle AOC = 2\angle AB'C = 2\beta$.

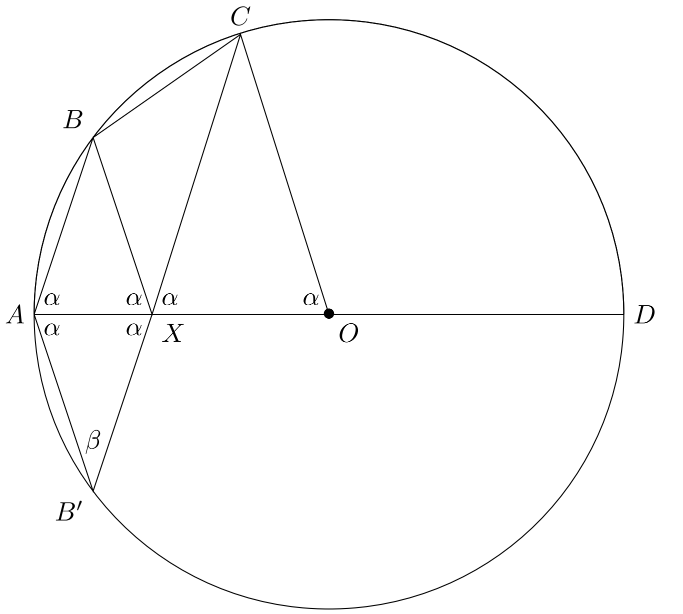
This gives $5\beta = 180°$, hence $\alpha = 72°$.

Method 4

Reflect triangle ABX in diameter AD to give triangle $AB'X$.
Note that $B'XC$ is a straight line.

Let $\angle AB'C = \beta$.
Note that $2\alpha + \beta = 180°$.

Draw BC.



Since AXO is a straight line, $\angle BXC = 180° - 2\alpha = \beta$.
In the cyclic quadrilateral $ABCB'$, $\angle BCB' + \angle BAB' = 180°$.
Hence $\angle BCB' = 180° - 2\alpha = \beta$. So triangle CBX is isosceles.

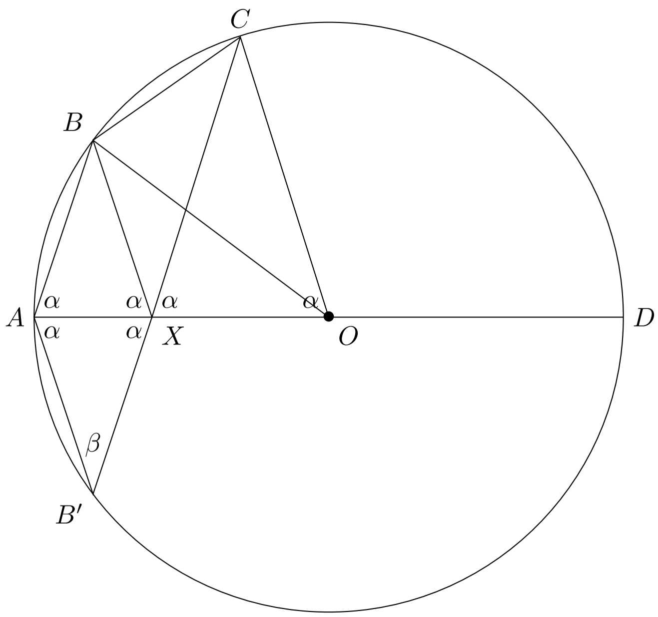
Hence $BC = BX = BA$. Also $OC = OA$.
This means $OABC$ is a kite and $\angle OAB = \angle OCB$.
So $\alpha = 2\beta$. This gives $5\beta = 180°$, hence $\alpha = 72°$.

Method 5

Reflect triangle ABX in diameter AD to give triangle $AB'X$.
Note that $B'XC$ is a straight line.

Let $\angle AB'C = \beta$.
Note that $2\alpha + \beta = 180^\circ$.

Draw BC and OB.



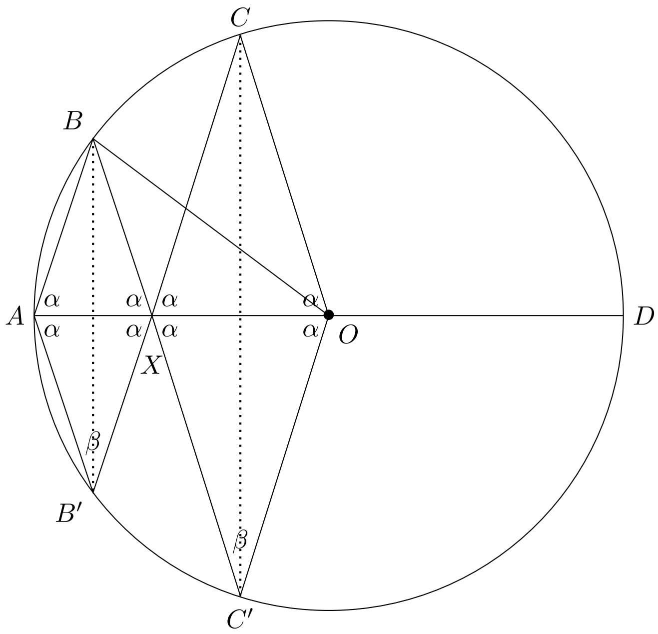
In isosceles triangle AOB, $\angle ABO = \alpha$, hence $\angle AOB = \beta$.
In cyclic quadrilateral $ABCB'$, $\angle ABC + \angle AB'C = 180^\circ$.
Hence $\angle OBC = (180 - \beta) - \alpha = \alpha$.

In isosceles triangle BOC, $\angle BCO = \alpha$, hence $\angle BOC = \beta$.
So $\alpha = \angle AOC = 2\beta$.
This gives $5\beta = 180^\circ$, hence $\alpha = 72^\circ$.

Method 6

Reflect triangles ABX, XCO in diameter AD to give triangles $AB'X$, $XC'O$.
Note that $B'XC$ and $C'XB$ are straight lines.

Draw lines BB' and CC'.
Let $\angle AB'C = \beta$. Then $\angle XC'O = \beta$.
Note that $2\alpha + \beta = 180°$.



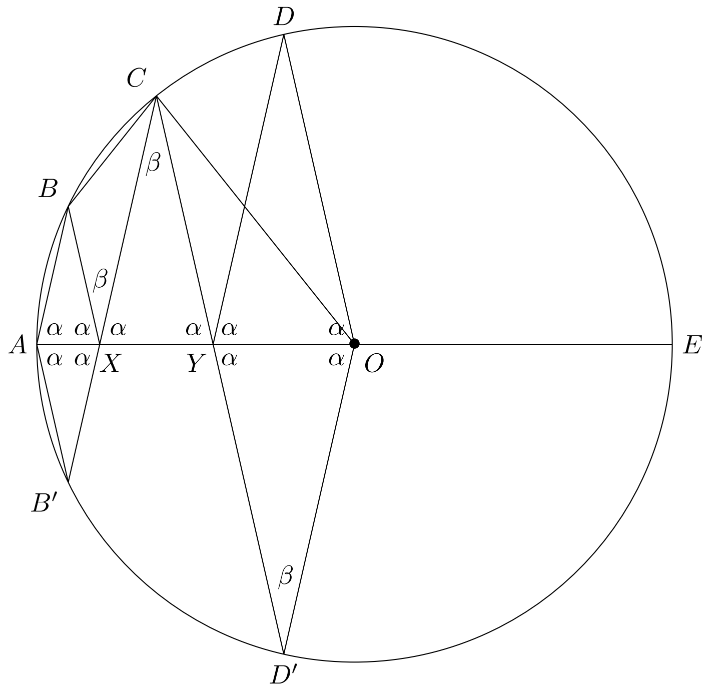
The two triangles that partition triangle BAB' are congruent (SAS).
Therefore BB' is perpendicular to AX, hence $\angle AB'B = \beta/2$.
Similarly, $\angle BC'C = \beta/2$.

Since the angle subtended by an arc at the centre is twice the angle subtended by the arc at the circumference, $\angle AOB = \beta$ and $\angle BOC = \beta$.
So $\alpha = \angle AOC = 2\beta$.
This gives $5\beta = 180°$, hence $\alpha = 72°$.

Investigation

(a) *Method 1*

Reflect the first and last triangles in the diameter and let $\angle XCY = \beta$.
Then $\angle OD'Y, \angle BXC = \beta$.
Draw BC and OC.

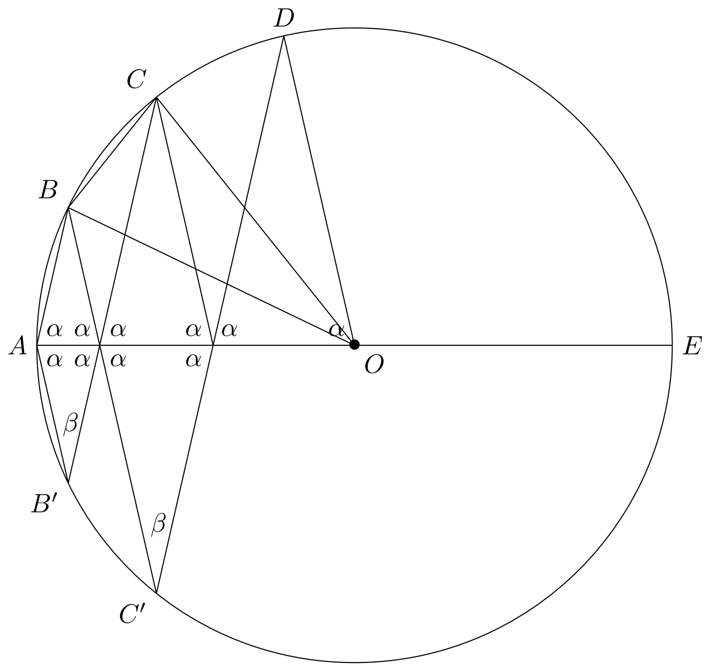


Since triangle COD' is isosceles, $\angle OCD' = \angle OD'C = \beta$.
As in solution method 4 of Question 10, $\angle BCB' = \beta$ and $OABC$ is a kite.
So $\alpha = \angle OCB = 3\beta$.

Since $2\alpha + \beta = 180°$, we have $\alpha = (3 \times 180)/7 = 540/7$ degrees.

Method 2

Reflect the first and second triangles in the diameter and let $\angle AB'C = \beta$.
Then $2\alpha + \beta = 180°$ and $\angle BC'D = \beta$.
Draw BC, CD, OB, OC.



In isosceles triangle AOB, $\angle ABO = \alpha$, hence $\angle AOB = \beta$.

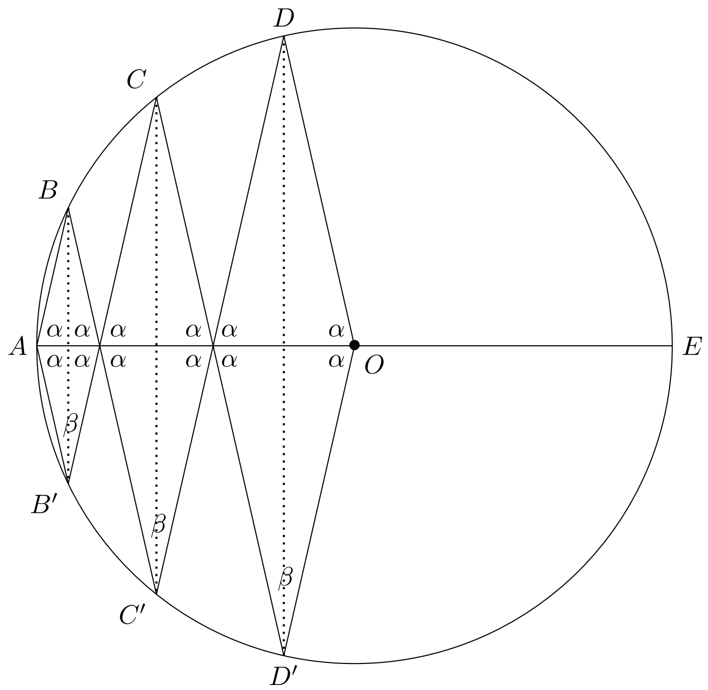
In cyclic quadrilateral $ABCB'$, $\angle ABC + \angle AB'C = 180°$.
Hence $\angle OBC = (180 - \beta) - \alpha = \alpha$.
In isosceles triangle BOC, $\angle BCO = \alpha$, hence $\angle BOC = \beta$.

In cyclic quadrilateral $BCDC'$, $\angle BCD + \angle BC'D = 180°$.
Hence $\angle OCD = (180 - \beta) - \alpha = \alpha$.
In isosceles triangle COD, $\angle CDO = \alpha$, hence $\angle COD = \beta$.

So $\alpha = \angle AOD = 3\beta$.
This gives $7\beta = 180°$, hence $\alpha = (3 \times 180)/7 = 540/7$ degrees.

Method 3

Reflect the three triangles in the diameter as shown and let $\angle AB'C = \beta$.
Then $2\alpha + \beta = 180°$ and angles $AB'C$, $BC'D$, $CD'O$ equal β.
Draw BB', CC', DD'.

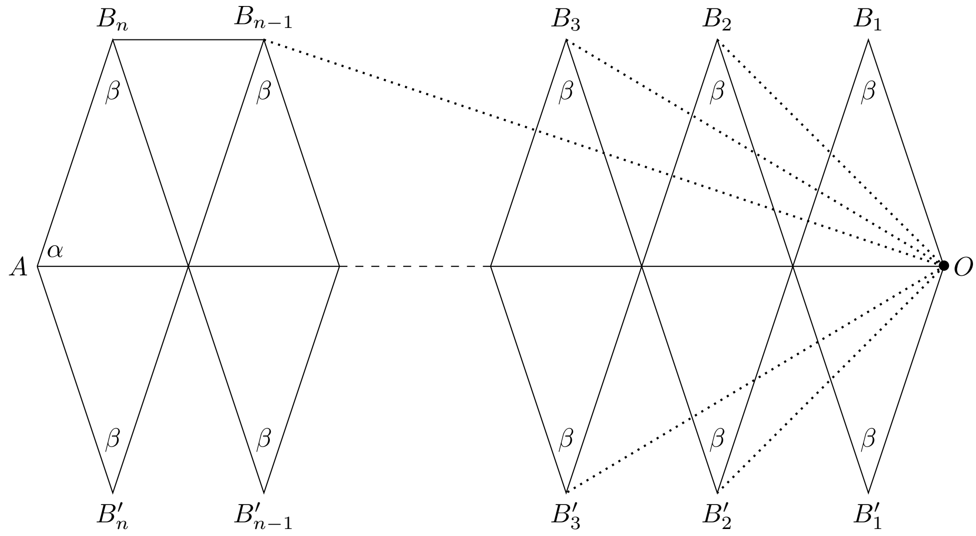


As in solution method 6 of Question 10, each of the angles $AB'B$, $BC'C$, $CD'D$ equals $\beta/2$. Since the angle subtended by an arc at the centre is twice the angle subtended by the arc at the circumference, each of the angles AOB, BOC, COD equals β.

So $\alpha = \angle AOD = 3\beta$.
This gives $7\beta = 180°$, hence $\alpha = (3 \times 180)/7 = 540/7$ degrees.

(b) *Method 1*

Consider this non-scale drawing of the n similar isosceles triangles and their reflection in AO. Draw the dotted radii as shown. Note again that $2\alpha + \beta = 180°$.



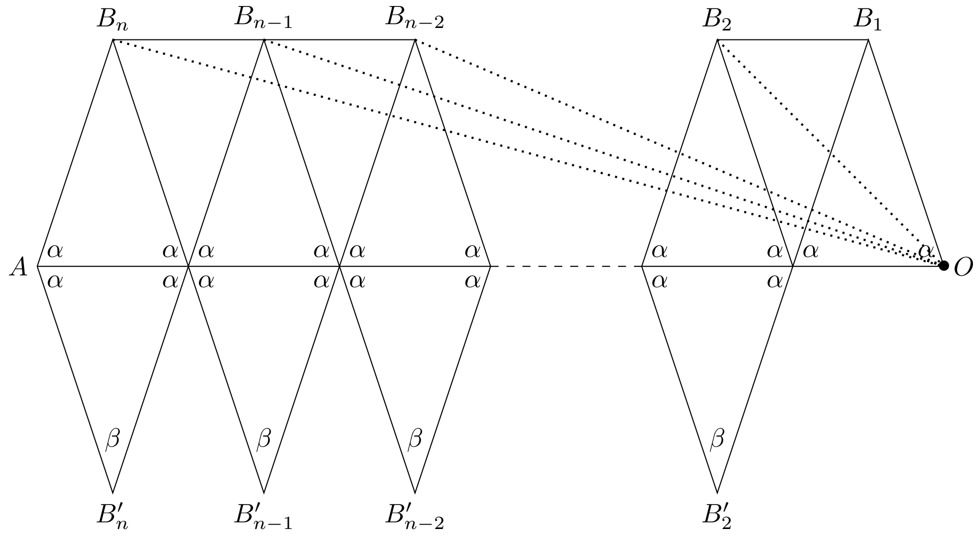
As in method 1 for Investigation (a), $\angle B_nB_{n-1}B'_n = \beta$ and OAB_nB_{n-1} is a kite.
We now calculate $\angle OB_{n-1}B'_n$.

From isosceles triangles, we have in succession the following observations:
$\angle OB_2B'_1 = \beta$ and $\angle OB'_2B_1 = \beta$, hence $\angle OB_2B'_3 = 2\beta = \angle OB'_2B_3$,
$\angle OB_3B'_2 = 2\beta$ and $\angle OB'_3B_2 = 2\beta$, hence $\angle OB_3B'_4 = 3\beta = \angle OB'_3B_4$,
and so on until we get $\angle OB_{n-1}B'_n = (n-1)\beta$.

So $\alpha = \beta + (n-1)\beta = n\beta$.
This gives $(2n+1)\beta = 180°$, hence $\alpha = 180n/(2n+1)$ degrees.

Method 2

Consider this non-scale drawing of the n similar isosceles triangles and their reflection in AO. Draw the dotted radii as shown. Note again that $2\alpha + \beta = 180°$.



In isosceles triangle AOB_n, $\angle AB_nO = \alpha$, hence $\angle AOB_n = \beta$.

In cyclic quadrilateral $AB_nB_{n-1}B'_n$, $\angle AB_nB_{n-1} + \angle AB'_nB_{n-1} = 180°$.
Hence $\angle OB_nB_{n-1} = (180 - \beta) - \alpha = \alpha$.
In isosceles triangle B_nOB_{n-1}, $\angle B_nB_{n-1}O = \alpha$, hence $\angle B_nOB_{n-1} = \beta$.

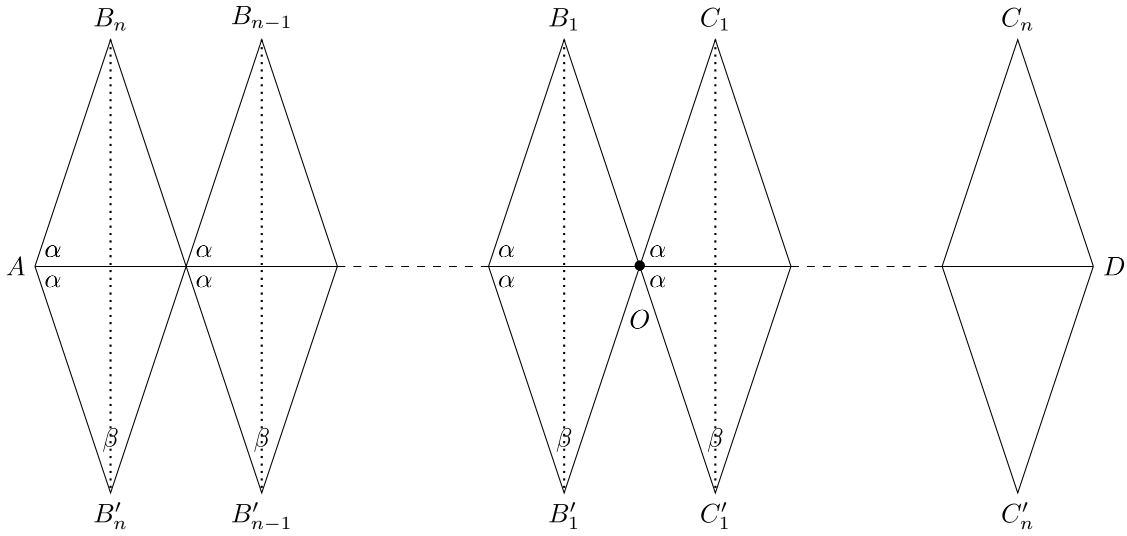
In cyclic quadrilateral $B_nB_{n-1}B_{n-2}B'_{n-1}$, $\angle B_nB'_{n-1}B_{n-2} + \angle B_nB_{n-1}B_{n-2} = 180°$.
Hence $\angle OB_{n-1}B_{n-2} = (180 - \beta) - \alpha = \alpha$.
In isosceles triangle $B_{n-1}OB_{n-2}$, $\angle B_{n-1}B_{n-2}O = \alpha$, hence $\angle B_{n-1}OB_{n-2} = \beta$.

Similarly, $\angle B_{k+1}OB_k = \beta$, for $k = n-3, n-4, \ldots, 1$.

So $\alpha = \angle AOB_1 = n\beta$.
This gives $(2n+1)\beta = 180°$, hence $\alpha = 180n/(2n+1)$ degrees.

Method 3

Consider this non-scale drawing of the n similar isosceles triangles, their reflection in AO, and the reflection of all $2n$ triangles in the line through O perpendicular to AO. Draw the dotted lines as shown. Note again that $2\alpha + \beta = 180°$.



As in method 3 for Investigation (a), each of the angles AB'_nB_n, $B_nB'_{n-1}B_{n-1}$, $\ldots$, $B_2B'_1B_1$, and B_1, C'_1C_1 equals $\beta/2$. Hence the angle at O subtended by each of the arcs AB_n, B_nB_{n-1}, $\ldots$, B_2B_1 and the arc B_1C_1 is β. By symmetry, all arcs defined by consecutive apexes of the $4n$ isosceles triangles and the points A and D subtend angle β at O. Therefore

$$\begin{aligned}
360 &= (4n+2)\beta \\
&= (4n+2)(180 - 2\alpha) \\
90 &= (2n+1)(90-\alpha) \\
&= 180n + 90 - (2n+1)\alpha \\
\alpha &= 180n/(2n+1).
\end{aligned}$$

Comments

1. From solution method 3 for Investigation (b), with $2n$ isosceles triangles on the diameter AD, their apexes plus reflections in AD and the points A and D are the vertices of a regular $(4n+2)$-sided polygon with internal angle $2\alpha = 180(1 - 1/(2n+1))$.

2. Similarly, if there are $2n-1$ isosceles triangles on the diameter AD, we get a regular $4n$-sided polygon with internal angle $2\alpha = 180(1 - 1/2n)$.

3. What about the converse? Does a regular polygon on a circle and with an even number of sides induce similar isosceles triangles whose bases partition a diameter through a pair of opposite vertices?

Australian Intermediate Mathematics Olympiad

Statistics

Distribution of Awards/School Year

Year	Number of Students	Number of Awards				
		Prize	High Distinction	Distinction	Credit	Participation
8	410	3	9	51	142	205
9	410	11	16	63	151	169
10	453	15	21	92	185	140
Other	266	2	5	17	74	168
All Years	**1539**	**31**	**51**	**223**	**552**	**682**

The award distribution is based on approximately the top 10% for High Distinction, next 15% for Distinction and the following 25% for Credit.

Number of Correct Answers Questions 1–8

Year	Number Correct / Question							
	1	2	3	4	5	6	7	8
8	350	213	176	271	97	139	38	39
9	361	254	192	297	124	186	52	47
10	405	291	203	327	155	256	73	81
Other	223	113	84	158	48	79	17	13
All Years	**1339**	**871**	**655**	**1053**	**424**	**660**	**180**	**180**

Mean Score/Question/School Year

School Year	Number of Students	Question			Overall Mean
		1–8	9	10	
8	410	10.2	0.3	0.4	10.4
9	410	11.8	0.6	0.6	12.5
10	453	12.8	0.7	0.7	13.8
Other	266	8.5	0.3	0.2	8.7
All Years	**1539**	**11.1**	**0.5**	**0.5**	**11.7**