QUESTIONS

1. A teacher asks Annie, Basil, Cara, Dave, and Elli, in that order, to each multiply a pair of (possibly equal) single digits. Except for Annie, each of them gets a product that is 50% more than the previous student. What product did *Dave* get?

[**2 marks**]

2. How many 4-digit palindromes are divisible by 99? (A *palindrome* is a number, not starting with 0, that is the same when its digits are written in reverse order.)

[**2 marks**]

3. Label the vertices of a square A, B, C, D anticlockwise. A line ℓ passes through B and intersects the side AD. If A is 5 cm from ℓ and C is 7 cm from ℓ, find the area of $ABCD$ in square centimetres.

[**3 marks**]

4. Three students Andrew, Louise and Elaine attempted 100 mathematics problems. Each of them solved exactly 60 problems. We call a problem *easy* if all three students solved it and *difficult* if only one student solved it. Given each problem was solved by at least one student, find the difference between the number of difficult problems and the number of easy problems.

[**3 marks**]

5. Fiona wants 6 red, 3 green and 3 yellow buttons on her dress. The buttons are indistinguishable except for colour. All buttons must be placed along a straight vertical line with no two neighbouring buttons of the same colour. How many ways could Fiona order the buttons?

[**3 marks**]

6. Three positive integers a, b, c with $a > c$ satisfy the following equations:

$$ac + b + c = bc + a + 66$$
$$a + b + c = 32$$

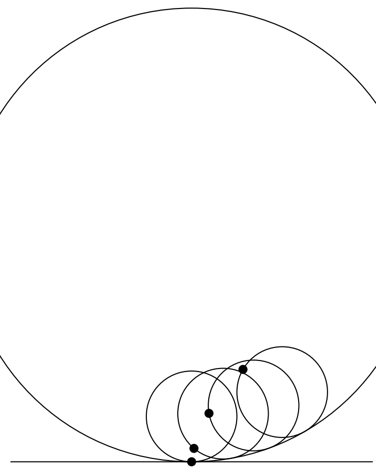
What is the value of abc?

[**4 marks**]

7. A positive integer is *positively assorted* if each of its digits are different and none are zero. Find the largest prime that divides the sum of all 4-digit positively assorted integers.

[**4 marks**]

8. A ride at an amusement park consists of a small circle which rotates inside a large circle. The large circle, of radius 100 metres, is tangent to the ground at its lowest point, and remains fixed in place throughout the ride. The smaller circle has radius 20 metres. It is initially tangent to the larger circle at its lowest point. Ada sits in the ride at this tangent point, on a seat attached to the smaller circle. When the ride starts, the smaller circle rolls around the larger circle without slipping, so that the circles always remain tangent, and Ada rotates with the smaller circle, as illustrated below.



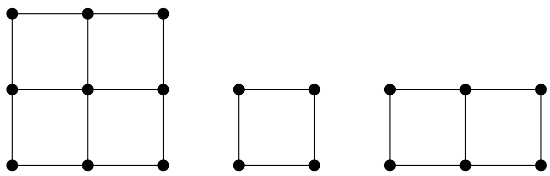
When the centre of the smaller circle has rotated 120° around the centre of the larger circle, how far in metres is Ada off the ground?

[**4 marks**]

9. Find all non-decreasing sequences of real numbers $a_1, a_2, a_3, \ldots$, such that $a_{2n} = 3a_n$ and $a_{3n} = 5a_n$ for all positive integers n.

[**5 marks**]

10. Let $n \geq m \geq 2$ and consider an $m \times n$ grid of unit squares. Such grids will contain many rectangles whose vertices are points on the grid, and whose edges are lines from the grid. For example, a 2×2 rectangular grid contains exactly three non-congruent rectangles as shown below.



Find all $m \times n$ grids that contain exactly 100 non-congruent rectangles.

[**5 marks**]

Investigation

Let $n \geq m \geq 2$. Describe the set of positive integers k for which there is an $m \times n$ grid containing k non-congruent rectangles.

[**3 bonus marks**]

1. We are looking for five products, which are all integers less than 100. If Annie's product is n, the fifth product will be $n \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = \frac{81n}{16}$. So n must be a multiple of 16.

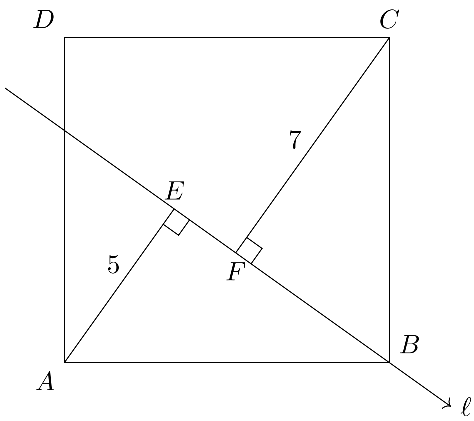
If $n = 16$, then the fifth product is 81. If n is any larger multiple of 16, the fifth product will be greater than 100. Hence $n = 16$ and Dave's product, is $16 \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = \mathbf{54}$.

2. A 4-digit palindrome has the form $\underline{abba}$ where a and b are digits. Since the sum of the first and third digits, $a + b$, is equal to the sum of the second and fourth digits, all 4-digit palindromes are multiples of 11.

So we seek numbers of the form $\underline{abba}$ which are divisible by 9. The sum of the digits is $2a + 2b = 2(a + b)$, so we seek $\{a, b\}$ such that $a + b = 9$ or 18. These pairs are {1,8}, {2,7}, {3,6}, {4,5}, {5,4}, {6,3}, {7,2}, {8,1}, {9,0}, {9,9}, giving the palindromes 1881, 2772, 3663, 4554, 5445, 6336, 7227, 8118, 9009, 9999, a total of **10**.

3. *Method 1*

Let E and F be the points shown.

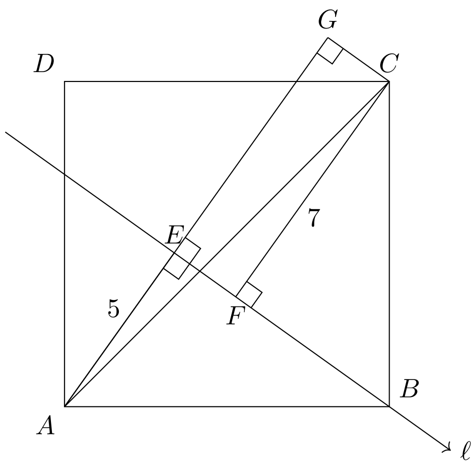


Since $\angle AEB$, $\angle BFC$, $\angle ABC$ are right angles, triangles AEB and BFC are equiangular with the same hypotenuse, hence congruent.

Therefore $EB = FC = 7$. From Pythagoras, $|ABCD| = AB^2 = AE^2 + EB^2 = 5^2 + 7^2 = \mathbf{74}$.

Method 2

Let E, F, G be the points shown.

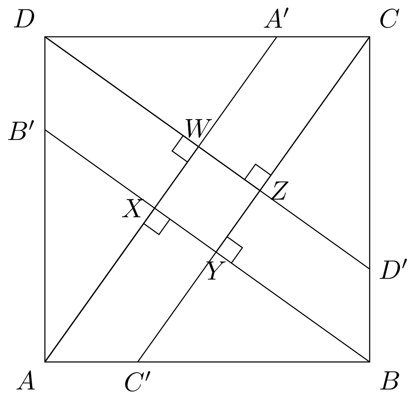


Since $\angle AEB$, $\angle BFC$, $\angle ABC$ are right angles, triangles AEB and BFC are equiangular with the same hypotenuse, hence congruent.

So $GC = EF = EB - FB = FC - EA = 7 - 5 = 2$, and $AG = AE + EG = AE + FC = 5 + 7 = 12$. From Pythagoras, $AB^2 + BC^2 = AC^2 = AG^2 + GC^2 = 144 + 4 = 148$. Since $AB = BC$, $|ABCD| = AB^2 = 148/2 = \mathbf{74}$.

Method 3

Let B' be the intersection of l and AD. Rotate the square $ABCD$ about its centre so that $\triangle ABB'$ becomes $\triangle BCC'$, then $\triangle CDD'$, and finally $\triangle DAA'$. Since each rotation is $90°$, the angles at the intersection points X, Y, Z, W are right angles.



Triangles AXB, BYC, CZD, DWA are equiangular with the same hypotenuse, hence congruent. Therefore, in particular, $AX = BY$ and $AW = BX = CY$.

Since $AX = 5$ and $CY = 7$, we have $XW = AW - AX = CY - AX = 7-5 = 2$ and $YX = BX - BY = CY - AX = 2$. Also, $|ABX| = \frac{1}{2} \times AX \times BX = \frac{1}{2} \times AX \times CY = \frac{1}{2} \times 5 \times 7$. So $|ABCD| = 4|ABX| + |XYZW| = 2 \times 35 + 4 = \mathbf{74}$.

4. *Method 1*

Let d be the number of difficult problems.
Let e be the number of easy problems.
Let t be the number of problems solved by exactly two of the students.

Each of the 100 problems is exactly one of these three types. This gives

$$d + e + t = 100$$

Now we count each problem each time a student solves it. This gives

$$d + 2t + 3e = 3 \times 60 = 180$$

Multiplying the first equation by 2 and subtracting the second gives $d - e = \mathbf{20}$.

Method 2

Number the students 1, 2, 3. Call a problem *medium* if it solved by exactly two students.

Let d_i be the number of difficult problems solved by student i.
Let m_i be the number of medium problems *not* solved by student i.
Let e be the number of easy problems.

Then

$$\begin{aligned} d_1 + m_2 + m_3 + e &= 60 \\ d_2 + m_1 + m_3 + e &= 60 \\ d_3 + m_1 + m_2 + e &= 60 \\ d_1 + d_2 + d_3 + m_1 + m_2 + m_3 + e &= 100 \end{aligned}$$

Multiplying the last equation by 2 and subtracting all the other equations gives
$d_1 + d_2 + d_3 - e = 200 - 180 = \mathbf{20}$.

5. *Method 1*

There are five spaces to fill between the six red buttons. Since there are 6 non-red buttons, there are at most two non-red buttons between any two otherwise adjacent red buttons.

Suppose there is exactly one non-red button between any two otherwise adjacent red buttons. Then there must be a red button at the top or at the bottom. The two cases are symmetrical. With a red button at the top, we must choose 3 out of 6 places for the three green buttons and then the three yellow buttons must occupy the remaining three places. The number of such arrangements is $\binom{6}{3} = 20$. So the number of arrangements with exactly one non-red button between any two otherwise adjacent red buttons is $20 + 20 = 40$.

Now suppose there are two non-red buttons between two otherwise adjacent red buttons. Then there are only 5 possible locations for the non-red pair and the remaining four non-red buttons must be placed separately in the remaining four places. The non-red pair must be green and yellow. They can be arranged in 2 ways and the number of arrangements for the two remaining green buttons is $\binom{4}{2} = 6$. Then the remaining two yellow buttons must occupy the remaining two places. So the number of arrangements with two non-red buttons between two otherwise adjacent red buttons is $5 \times 2 \times 6 = 60$.

Thus the total number of ways Fiona can arrange the buttons on her dress is $40 + 60 = \mathbf{100}$.

Method 2

Each non-red button must be placed in one of 7 places: the 5 places between the 6 red buttons plus the 2 places above and below all the red buttons. Initially, we suppose the 6 non-red buttons are distinguishable and count relevant permutations.

Suppose there is at most one non-red button between any two otherwise adjacent red buttons, then we have 7 objects to permute in the available 7 places: the 6 non-red buttons and a 'non-button'. There are 7! such permutations. Of these, we must delete the permutations that have the non-button placed in the 5 places between the red buttons. There are $5 \times 6!$ such permutations. So the number of legitimate permutations in this case is $7! - 5 \times 6! = 2 \times 6!$.

Now suppose there are two non-red buttons between two otherwise adjacent red buttons. Then there are only 5 possible locations for the non-red pair and the remaining four non-red buttons must be placed separately in the remaining four places between the red buttons. At each location for the non-red pair, we have 6 choices of buttons for the first in the pair, then 3 choices for the second. So the number of legitimate permutations in this case is $5 \times 6 \times 3 \times 4! = 3 \times 6!$.

Thus the total number of legitimate permutations of the 6 non-red buttons is $5 \times 6!$.

Let p be the number of legitimate placements of the 6 non-red buttons. Since they consist of 3 indistinguishable green buttons and 3 indistinguishable yellow buttons, we have $p3!3! = 5 \times 6!$. Hence $p = (5 \times 6!)/(3! \times 3!) = \mathbf{100}$.

6. Re-arranging the first equation gives:

$$65 = ac - bc + c - a + b - 1 = (a - b + 1)(c - 1)$$

We know a, b, c are integers, so $a - b + 1$ and $c - 1$ are factors of 65: 1, 5, 13, 65.
So c is one of 2, 6, 14, 66, and $a - b$ is 64, 12, 4, 0 respectively.

If $c = 66$, then the second given equation gives $a + b = -34$, which contradicts the requirement that all three integers are positive.
If $c = 2$, then $a + b = 30$ and $a - b = 64$, giving $b = -17$ again contradicting positivity.
If $c = 14$, then $a + b = 18$ and $a - b = 4$, giving $a = 11$, which contradicts $a > c$.

This leaves $c = 6$, which gives $a + b = 26$ and $a - b = 12$.
Hence $a = 19$, $b = 7$, and $abc = 19 \times 7 \times 6 = \mathbf{798}$.

7. *Method 1*

We first enumerate the 4-digit positively assorted integers. There are 9 choices for the first digit, each of which leaves 8 choices for the second, then 7 choices for the third, and 6 choices for the fourth digit. So we have $9 \times 8 \times 7 \times 6$ positively assorted integers with four digits.

We pair each 4-digit positively assorted integer $\underline{abcd}$ with another 4-digit positively assorted integer $\underline{wxyz}$, where

$$w = 10 - a, x = 10 - b, y = 10 - c, z = 10 - d$$

The sum of each pair is

$$1000(a + 10 - a) + 100(b + 10 - b) + 10(c + 10 - c) + (d + 10 - d) = 11110$$

Therefore, the sum of all 4-digit positively assorted integers is

$$\tfrac{1}{2}(9 \times 8 \times 7 \times 6) \times 11110 = 2^4 \times 3^3 \times 7 \times 5555 = 2^4 \times 3^3 \times 5 \times 7 \times 11 \times 101$$

So the largest prime divisor of this sum is **101**.

Method 2

We first enumerate the 4-digit positively assorted integers that have 1 as the first digit. There are 8 choices for the second digit, each of which leaves 7 choices for the third, then 6 choices for the fourth digit. So there are $8 \times 7 \times 6$ positively assorted 4-digit integers with 1 as the first digit.

Therefore, when all 4-digit positively assorted integers with 1 as the first digit are added, digit 1 will contribute $1 \times 10^3 \times 8 \times 7 \times 6$. Similarly, when all 4-digit positively assorted integers with d as the first digit are added, digit d will contribute $d \times 10^3 \times 8 \times 7 \times 6$.

So, when all 4-digit positively assorted integers are added, the first digits will contribute

$$(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) \times 10^3 \times 8 \times 7 \times 6 = 45 \times 10^3 \times 8 \times 7 \times 6$$

Similarly, when all 4-digit positively assorted integers are added, the second digits will contribute $45 \times 10^2 \times 8 \times 7 \times 6$, the third digits will contribute $45 \times 10 \times 8 \times 7 \times 6$, and the fourth digits will contribute $45 \times 1 \times 8 \times 7 \times 6$.

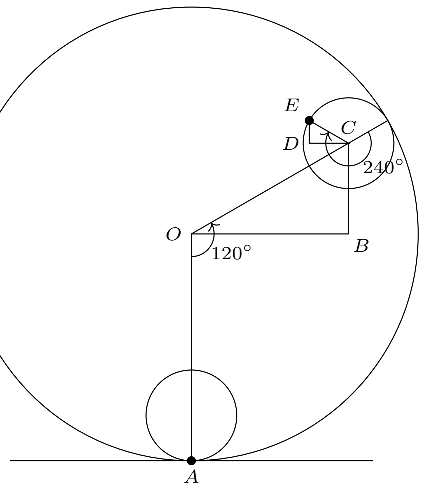
Therefore, the sum of all 4-digit positively assorted integers is

$$45 \times (1 + 10 + 100 + 1000) \times 8 \times 7 \times 6 = 45 \times 1111 \times 8 \times 7 \times 6 = 45 \times 11 \times 101 \times 8 \times 7 \times 6$$

So the largest prime divisor of this sum is **101**.

8. *Method 1*

Ada's final and initial positions are shown below. The centre of the larger circle is denoted O; the final position of the centre of smaller circle is denoted C; Ada's final position is denoted E. We drop a vertical line from E, denoting the point at the same height as C by D. Similarly we drop a vertical line from C, denoting the point at the same height as O by B.



The arc traversed on the larger circle has length $100 \times \frac{2\pi}{3} = \frac{200\pi}{3}$. The circumference of the smaller circle is 40π. Since the smaller circle rotates without slipping, Ada's seat has rotated about the smaller circle's centre, and relative to the point of tangency of the two circles, through an angle of $360 \times \frac{200\pi}{3}/40\pi = 120 \times 5 = 600°$. As $600 = 360 + 240$, this is one full revolution, plus a further 240°. Hence Ada's final position is 240° clockwise from the point of tangency of the two circles, as shown.

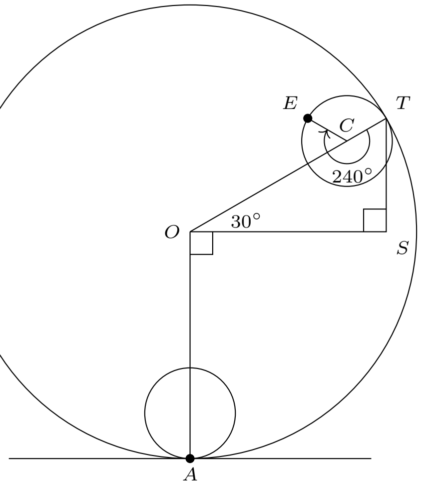
Ada's height off the ground is $AO + BC + DE$. Now AO is the radius of the larger circle, which is 100 m. Since $\angle BOC = 120 - 90 = 30°$, triangle BOC has angles $30°, 60°, 90°$. Its hypotenuse OC is the difference between the radii of the circles, hence $100 - 20 = 80$. So, noting that BOC is half an equilateral triangle of side length 80, we observe $BC = 40$ m.

Since OB and CD are parallel, $\angle DCO = \angle COB = 30°$. So $\angle DCE = 240 - 30 - 180 = 30°$. Thus triangle CDE has angles $30°, 60°, 90°$ and its hypotenuse is the radius of the smaller circle, 20. Again, recognising half an equilateral triangle, we obtain $DE = CE/2 = 10$.

Thus Ada's height off the ground, in metres, is $AO + BC + DE = 100 + 40 + 10 = \mathbf{150}$.

Method 2

Ada's final and initial positions are shown below. The centre of the larger circle is denoted O; the final position of the centre of smaller circle is denoted C; Ada's final position is denoted E; the point of tangency of the two circles is denoted T. In $\triangle OST$, OS is horizontal and ST is vertical. So $\angle SOT = 120 - 90 = 30°$.



The arc traversed on the larger circle has length $100 \times \frac{2\pi}{3} = \frac{200\pi}{3}$. The circumference of the smaller circle is 40π. Since the smaller circle rotates without slipping, Ada's seat has rotated about the smaller circle's centre, and relative to the point of tangency of the two circles, through an angle of $360 \times \frac{200\pi}{3}/40\pi = 120 \times 5 = 600°$. As $600 = 360 + 240$, this is one full revolution, plus a further 240°. Hence Ada's final position is 240° clockwise from T, as shown.

Since CT is at 30° to the horizontal, the angle at which EC is above the horizontal is $240 - 180 - 30 = 30°$. So E and T are at the same height above the ground.

Since $\triangle OST$ has angles $30°, 90°, 60°$, it is half an equilateral triangle with side length 100, hence $ST = OT/2 = 50\,\text{m}$. So Ada's height above the ground, in metres, is $TS + OA = 50 + 100 = \mathbf{150}$.

Comment

The curve that Ada follows is called a *hypocycloid*. Students who know some trigonometry might know the parametric equations for a hypocycloid.

Let R be the radius of the large circle and r the radius of the small circle. Place the centre of the large circle at the origin of Cartesian axes and start Ada at $(R, 0)$. Let θ be the angle from the positive x-axis to the line through the centres of both circles. Then the Cartesian coordinates for Ada are:

$$x = (R - r)\cos\theta + r\cos(\tfrac{R-r}{r}\theta)$$
$$y = (R - r)\sin\theta - r\sin(\tfrac{R-r}{r}\theta)$$

In the present problem, we have $R = 100$, $r = 20$, and we want the x-coordinate when $\theta = 120°$.
So $x = 80\cos 120 + 20\cos(4 \times 120) = 80\cos 120 + 20\cos 120 = 100\cos 120 = -50$.
Hence the required height for Ada is $100 + 50 = 150$.

9. *Method 1*

We show that the only such sequence is the sequence of all zeroes. First we show $a_1 = 0$.

We have $a_2 \le a_3$. Also $a_2 = 3a_1$, and $a_3 = 5a_1$.
Putting these together gives $3a_1 \le 5a_1$, so $a_1 \ge 0$.

On the other hand, we have $a_8 \le a_9$. Also $a_8 = 3a_4 = 9a_2 = 27a_1$, and $a_9 = 5a_3 = 25a_1$.
Putting these together gives $27a_1 \le 25a_1$, so $a_1 \le 0$.

Since $a_1 \ge 0$ and $a_1 \le 0$, we conclude $a_1 = 0$.

Now we show $a_n = 0$ for all positive integers n. If n is a power of 2, then for some positive integer k we have $a_n = a_{2^k} = 3^k a_1 = 0$.

If n is between consecutive powers of 2, then for some positive integer k we have $2^k < n < 2^{k+1}$.
So $0 = a_{2^k} \le a_n \le a_{2^{k+1}} = 0$. Hence $a_n = 0$.

Method 2

We show directly that $a_k = 0$ for any positive integer k.

We have $a_{8k} = 3a_{4k} = 9a_{2k} = 27a_k$ and $a_{9k} = 5a_{3k} = 25a_k$.

Since $a_k \le a_{8k} \le a_{9k}$, we have $a_k \le 27a_k \le 25a_k$.

The first inequality yields $a_k \ge 0$. The second yields $a_k \le 0$.

Hence $a_k = 0$.

10. There are n non-congruent rectangles of the form $1 \times r$: $1 \times 1, 1 \times 2, \ldots, 1 \times n$.

There are $n-1$ non-congruent rectangles of the form $2 \times r$ with $r \ge 2$: $2 \times 2, 2 \times 3, \ldots, 2 \times n$.

In general, with $1 \le k \le m$, there are $n-(k-1)$ non-congruent rectangles of the form $k \times r$ with $r \ge k$: $k \times k, k \times (k+1), \ldots, k \times n$.

Therefore the total number of non-congruent rectangles in an $m \times n$ grid is

$$n + (n-1) + \cdots + (n-m+1) = \tfrac{1}{2}m(2n-m+1)$$

We want $\frac{1}{2}m(2n-m+1) = 100$, that is, $m(2n-m+1) = 200$.

Method 1

Since $2n-m+1 \ge 2m-m+1 > m$, m is one of 2, 4, 5, 8, 10.

If $m = 2$, then $2n-1 = 100$, which is impossible.
If $m = 4$, then $2n-3 = 50$, which is impossible.
If $m = 5$, then $2n-4 = 40$, and $n = 22$.
If $m = 8$, then $2n-7 = 25$, and $n = 16$.
If $m = 10$, then $2n-9 = 20$, which is impossible.

So the only grids that contain exactly 100 non-congruent rectangles are 5×22 and 8×16.

Method 2

We have this quadratic in m: $m^2 - (2n+1)m + 200 = 0$. Since $2n+1$ is odd, one value of m is odd and the other is even. The product of both is 200.

So we have $(m-5)(m-40) = 0$ or $(m-25)(m-8) = 0$. In the first case, $2n+1 = 45$ and $n = 22 < 40$, so $m = 5$. In the second case, $2n+1 = 33$ and $n = 16 < 25$, so $m = 8$.

So the only grids that contain exactly 100 non-congruent rectangles are 5×22 and 8×16.

Investigation

We will prove that there is an $m \times n$ grid with k non-congruent rectangles if and only if k is not a power of 2.

First, suppose that k has an odd factor, q. Let $k = qp$. We want to find integer values $n \ge m \ge 2$ for which $\frac{1}{2}m(2n-m+1) = qp$. Indeed, if we let $m = 2p$ and $n = (q+2p-1)/2$ then

$$\tfrac{1}{2}m(2n-m+1) = \tfrac{1}{2}(2p)(q+2p-1-2p+1) = pq$$

Conversely, suppose that k is a power of 2.
Then $\frac{1}{2}m(2n-m+1) = 2^r$ for some integer r. So $m(2n-m+1) = 2^{r+1}$.

Since $m > 1$, m must be even. Then $2n-m+1$ is odd.
This means $2n-m+1 = 1$, hence $2n = m$. But this contradicts $m \ge 2$.

Statistics

Distribution of Awards/School Year

Year	Number of Students	Number of Awards				
		Prize	High Distinction	Distinction	Credit	Participation
8	350	2	7	46	119	174
9	414	11	11	82	151	156
10	338	12	22	77	109	118
Other	513	7	18	76	146	261
All Years	**1615**	**32**	**58**	**281**	**525**	**709**

The award distribution is based on approximately the top 10% for High Distinction, next 15% for Distinction and the following 25% for Credit.

Number of Correct Answers Questions 1–8

Year	Number Correct / Question							
	1	2	3	4	5	6	7	8
8	293	216	91	284	32	141	73	84
9	346	256	147	340	68	200	130	117
10	288	190	152	286	66	160	124	122
Other	373	269	170	405	55	209	132	106
All Years	**1300**	**931**	**560**	**1315**	**221**	**710**	**459**	**429**

Mean Score/Question/School Year

School Year	Number of Students	Question			Overall Mean
		1–8	9	10	
8	350	11.2	0.3	0.8	11.9
9	414	13.0	0.8	1.1	13.9
10	338	13.8	0.9	1.5	15.7
Other	513	11.3	0.6	1.1	12.4
All Years	**1615**	**12.2**	**0.6**	**1.1**	**13.3**