Questions

Questions 1 to 8 only require their numerical answers all of which are non-negative integers less than 1000.
Questions 9 and 10 require written solutions which may include proofs.
The bonus marks for the Investigation in Question 10 may be used to determine prize winners.

1. A 3-digit number $\underline{abc}$ is multiplied by 3 to give the 4-digit number $\underline{c0ba}$. Find the number $\underline{abc}$. [**2 marks**]

2. A point D lies on the side AC of a triangle ABC. Triangle ADB is isosceles with $DA = DB$. Triangle DBC is also isosceles with $BC = BD$. All angles in triangles ADB and DBC are an integer number of degrees. What is the difference between the largest and smallest values that angle ADB could have? [**2 marks**]

3. Amy has 14 cousins aged 2, 3, 4, ..., 15 respectively. She also has some cards separately numbered 16, 17, 18, ..., k, for some integer k. Amy manages to give one card to each cousin so that the number on the card is a multiple of that cousin's age. Find the least k for which this is possible. [**3 marks**]

4. If $3^x - 3^{-x} = \sqrt{285}$, what is $3^x + 3^{-x}$? [**3 marks**]

5. There are 5 lily pads on a pond, arranged in a circle. A frog can only jump from each lily pad to an adjacent lily pad on either side. How many ways are there for the frog to start on one of these lily pads, make 11 jumps, and end up where it started? [**3 marks**]

6. Find the sum of *all* (not necessarily distinct) values of a over *all* triples (a, b, c) of real numbers that satisfy the equations:

$$(a+b)(c+1) = 22 \qquad (1)$$
$$(a+c)(b+1) = 22 \qquad (2)$$
$$(b+c)(a+1) = 22 \qquad (3)$$

[**4 marks**]

7. A triangle has sides of length $x, y, 20$ where $x > y > 20$ and x and y are integers. Let h be the length of the altitude of the triangle from the side of length 20. Let h_x and h_y be the lengths of the altitudes from the sides of length x and y respectively. These altitudes are such that $h = h_x + h_y$. Find the perimeter of this triangle. [**4 marks**]

8. A *word* is a sequence of zero or more letters taken from the set {A, B, C, D, E, F, G, H, I}. Two words are said to be *related* if one can be obtained from the other by a sequence of the following operations:
(a) swapping two adjacent letters,
(b) deleting two adjacent letters which are the same,
(c) inserting two adjacent letters which are the same.

Thus, for instance, the words BACA and BCAA are related by swapping adjacent letters A and C, and BCAA is related to BC by deleting or inserting two adjacent letter As. Note that the empty word with zero letters is related to any word consisting of just two adjacent letters.

What is the maximum number of words in a set in which no word is related to any other? [**4 marks**]

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9. Find all positive integers n for which $9^{9^n} + 91^{91^n}$ is divisible by 100. [**5 marks**]

10. Identical wind turbines are equally spaced in a straight line on level ground. Each turbine tower is a vertical cylinder of radius 1 metre. Let $\mathcal{L}$ be the line through the base centres of the towers. There is an observer fixed at O on the same level ground as the towers. Let A be the point on $\mathcal{L}$ that is closest to O.

If A is the base centre of a tower and $OA = 15$ metres, what is the maximum number of towers that are completely visible to the observer over all possible distances d metres between the base centres of adjacent towers?
[**5 marks**]

Investigation

If $OA = 16$ metres and A is *not* the base centre of a tower, what is the maximum number of towers that are completely visible to the observer? [**4 bonus marks**]

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