Questions

Questions 1 to 8 only require their numerical answers all of which are non-negative integers less than 1000.
Questions 9 and 10 require written solutions which may include proofs.
The bonus marks for the Investigation in Question 10 may be used to determine prize winners.

1. A 3-digit number $\underline{abc}$ is multiplied by 3 to give the 4-digit number $\underline{c0ba}$. Find the number $\underline{abc}$. [**2 marks**]

2. A point D lies on the side AC of a triangle ABC. Triangle ADB is isosceles with $DA = DB$. Triangle DBC is also isosceles with $BC = BD$. All angles in triangles ADB and DBC are an integer number of degrees. What is the difference between the largest and smallest values that angle ADB could have? [**2 marks**]

3. Amy has 14 cousins aged 2, 3, 4, …, 15 respectively. She also has some cards separately numbered 16, 17, 18, …, k, for some integer k. Amy manages to give one card to each cousin so that the number on the card is a multiple of that cousin's age. Find the least k for which this is possible. [**3 marks**]

4. If $3^x - 3^{-x} = \sqrt{285}$, what is $3^x + 3^{-x}$? [**3 marks**]

5. There are 5 lily pads on a pond, arranged in a circle. A frog can only jump from each lily pad to an adjacent lily pad on either side. How many ways are there for the frog to start on one of these lily pads, make 11 jumps, and end up where it started? [**3 marks**]

6. Find the sum of *all* (not necessarily distinct) values of a over *all* triples (a, b, c) of real numbers that satisfy the equations:

$$(a+b)(c+1) = 22 \qquad (1)$$
$$(a+c)(b+1) = 22 \qquad (2)$$
$$(b+c)(a+1) = 22 \qquad (3)$$

[**4 marks**]

7. A triangle has sides of length $x, y, 20$ where $x > y > 20$ and x and y are integers. Let h be the length of the altitude of the triangle from the side of length 20. Let h_x and h_y be the lengths of the altitudes from the sides of length x and y respectively. These altitudes are such that $h = h_x + h_y$. Find the perimeter of this triangle. [**4 marks**]

8. A *word* is a sequence of zero or more letters taken from the set {A, B, C, D, E, F, G, H, I}. Two words are said to be *related* if one can be obtained from the other by a sequence of the following operations:
(a) swapping two adjacent letters,
(b) deleting two adjacent letters which are the same,
(c) inserting two adjacent letters which are the same.

Thus, for instance, the words BACA and BCAA are related by swapping adjacent letters A and C, and BCAA is related to BC by deleting or inserting two adjacent letter As. Note that the empty word with zero letters is related to any word consisting of just two adjacent letters.

What is the maximum number of words in a set in which no word is related to any other? [**4 marks**]

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9. Find all positive integers n for which $9^{9^n} + 91^{91^n}$ is divisible by 100. [**5 marks**]

10. Identical wind turbines are equally spaced in a straight line on level ground. Each turbine tower is a vertical cylinder of radius 1 metre. Let $\mathcal{L}$ be the line through the base centres of the towers. There is an observer fixed at O on the same level ground as the towers. Let A be the point on $\mathcal{L}$ that is closest to O.

If A is the base centre of a tower and $OA = 15$ metres, what is the maximum number of towers that are completely visible to the observer over all possible distances d metres between the base centres of adjacent towers?

[**5 marks**]

Investigation

If $OA = 16$ metres and A is *not* the base centre of a tower, what is the maximum number of towers that are completely visible to the observer? [**4 bonus marks**]

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Solutions

1. *Method 1*

When a 3-digit number is multiplied by 3, it cannot be larger than 2997. Hence c is either 1 or 2. It follows that $a = 3c$ as there would be no carry in the multiplication.

We have $3(100a + 10b + c) = 1000c + 10b + a$. Substituting $a = 3c$ gives $903c + 30b = 1003c + 10b$ which simplifies to $b = 5c$. Since $b < 10$, the only solution is $c = 1, b = 5, a = 3$.

Therefore $\underline{abc} = \mathbf{351}$.

Method 2

When a 3-digit number is multiplied by 3, it cannot be larger than 2997. Hence c is either 1 or 2. It follows that $a = 3c$ as there would be no carry in the multiplication.

The last digit of $3b$ is b. Hence $b = 0$ or 5.
If $b = 0$, we have $3 \times \underline{abc} = 3 \times 301 = 903 \neq 1003$, or $3 \times \underline{abc} = 3 \times 602 = 1806 \neq 2006$.
If $b = 5$, we have $3 \times \underline{abc} = 3 \times 351 = 1053 = \underline{c0ba}$, or $3 \times \underline{abc} = 3 \times 652 = 1956 \neq 2056$.

Therefore $\underline{abc} = \mathbf{351}$.

Method 3

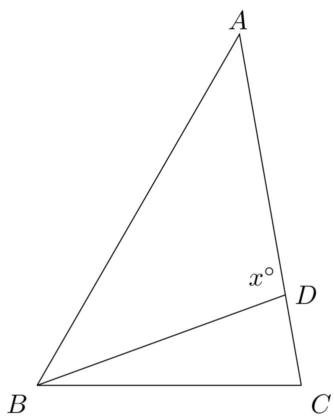
We have
$$\begin{aligned} 3(100a + 10b + c) &= 1000c + 10b + a \\ 20b &= 1000c - 3c + a - 300a \\ b &= 50c - 15a + (a - 3c)/20 \end{aligned}$$

Since $1 \leq a, c \leq 9$, we have $-26 \leq a - 3c \leq 6$, hence $a - 3c = 0$ or -20.
If $a - 3c = -20$, then $b = 50c - 15(3c - 20) - 1 = 5c + 299 \geq 304$, which contradicts $b \leq 9$.
If $a - 3c = 0$, then $b = 50c - 45c = 5c$. Since b and c are digits, $c = 1$, $b = 5$, and $a = 3$.

Therefore $\underline{abc} = \mathbf{351}$.

2. *Method 1*

Let $\angle ADB = x^\circ$.



Thus $x < 180$. Since $\angle BAD = 90 - \frac{x}{2}$, x is even. So $x \leq 178$.

Since $\angle CBD = 180 - 2(180 - x) = 2x - 180$, $2x > 180$. So $x \geq 92$.

All angles are integers if $x = 92$ and all angles are integers if $x = 178$.
Hence the difference between the largest and smallest values of $\angle ADB$ is $178 - 92 = \mathbf{86}$.

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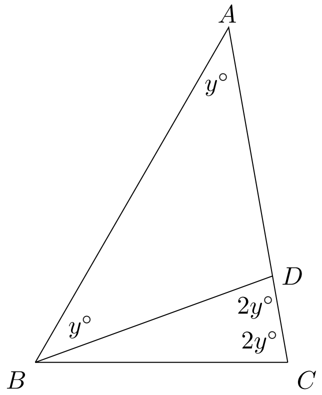
Method 2

Let $\angle BAD = y°$. Then we have



Since $\angle CBD = 180 - 4y > 0$, we have $y < 45$, hence $1 \leq y \leq 44$.

So $\angle ADB = 180 - 2y \geq 180 - 88 = 92$ and $\angle ADB \leq 180 - 2 = 178$.

All angles are integers if $\angle ADB = 92$ and all angles are integers if $\angle ADB = 178$.
Hence the difference between the largest and smallest values of $\angle ADB$ is $178 - 92 = \mathbf{86}$.

3. We want at least 14 composite numbers in the range 16 to k. The first 14 composites from 16 are 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, 32, 33, 34. Therefore $k \geq 34$.

Working from $m = 15$ down to $m = 2$, the table shows all multiples of m that can be selected from the range above and have not been selected for previous values of m.

m	multiples	m	multiples	m	multiples	m	multiples
15	30	11	22, 33	7	21	3	27, 33
14	28	10	20	6	18	2	16, 22, 32, 34
13	26	9	18, 27	5	25		
12	24	8	16, 32	4	16, 32		

The only allowable multiple for 6 is 18, hence 27 for 9, 33 for 3, and 22 for 11. Choosing 16 as the multiple for 8, leaves only 32 for 4, and 34 for 2. So we have 14 acceptable multiples:
30, 28, 26, 24, 22, 20, 27, 16, 21, 18, 25, 32, 33, 34.

Therefore the minimum value of k is **34**.

Comment

Choosing 32 as the multiple for 8 gives the only other acceptable distribution from the 14 cards:
30, 28, 26, 24, 22, 20, 27, 32, 21, 18, 25, 16, 33, 34.

4. *Method 1*

Observe that $285 = (3^x - 3^{-x})^2 = 3^{2x} + 3^{-2x} - 2$.

Also $(3^x + 3^{-x})^2 = 3^{2x} + 3^{-2x} + 2 = 285 + 4 = 289$.

Since 3^x and 3^{-x} are positive, we have $3^x + 3^{-x} = \sqrt{289} = \mathbf{17}$.

Method 2

Let $3^x + 3^{-x} = a$.
Adding $3^x - 3^{-x} = \sqrt{285}$ gives $2 \times 3^x = a + \sqrt{285}$.

Subtracting $3^x - 3^{-x} = \sqrt{285}$ gives $2 \times 3^{-x} = a - \sqrt{285}$.

Multiplying gives $4 = a^2 - 285$. Since $a > 0$, $a = \sqrt{289} = \mathbf{17}$.

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Method 3

Let $t = 3^x$, then $t - 1/t = \sqrt{285}$, hence $t^2 - \sqrt{285}t - 1 = 0$.

Since $3^x > 0$, we have
$$\begin{aligned} t &= \tfrac{1}{2}(\sqrt{285} + \sqrt{285 + 4}) \\ &= \tfrac{1}{2}(\sqrt{285} + \sqrt{289}) \\ &= \tfrac{1}{2}(\sqrt{285} + 17) \end{aligned}$$

So
$$\begin{aligned} 3^x + 3^{-x} &= \tfrac{1}{2}(\sqrt{285} + 17) + \tfrac{2}{\sqrt{285}+17} \\ &= \tfrac{1}{2}(\sqrt{285} + 17) + \tfrac{2(\sqrt{285}-17)}{(\sqrt{285}+17)(\sqrt{285}-17)} \\ &= \tfrac{1}{2}(\sqrt{285} + 17) + \tfrac{2(\sqrt{285}-17)}{285-17^2} \\ &= \tfrac{1}{2}(\sqrt{285} + 17) - \tfrac{1}{2}(\sqrt{285} - 17) \\ &= \mathbf{17} \end{aligned}$$

Method 4

Let $t = 3^x$, then $t - 1/t = \sqrt{285}$, hence $t^2 - \sqrt{285}t - 1 = 0$.

The roots of this quadratic are $\frac{1}{2}(\sqrt{285} \pm \sqrt{285 + 4}) = \frac{1}{2}(\sqrt{285} \pm \sqrt{289}) = \frac{1}{2}(\sqrt{285} \pm 17)$.

Since the product of these roots is -1 (the constant coefficient), the roots are t and $-t^{-1}$.
Hence $3^x + 3^{-x} = t + t^{-1} = \frac{1}{2}(\sqrt{285} + 17) - \frac{1}{2}(\sqrt{285} - 17) = \mathbf{17}$.

5. *Method 1*

Suppose, in some order, the frog makes c clockwise jumps and $11 - c$ anticlockwise jumps. The net number of anticlockwise jumps is $11 - 2c$. Hence, the frog ends where it started if and only if $11 - 2c$ is divisible by 5. Since $0 \le c \le 11$, the only values of c are 3 and 8.

Suppose the frog makes 3 clockwise and 8 anticlockwise jumps. Let k be the number of jumps between the first and third clockwise jumps. For a given value of $k = 1, 2, 3, \ldots, 9$, the first and third clockwise jumps could occupy $10 - k$ pairs of positions. For each of these positions, the second clockwise jump could occupy k positions. So the number of ways the frog could make 3 clockwise and 8 anticlockwise jumps is

$$(9 \times 1) + (8 \times 2) + (7 \times 3) + (6 \times 4) + (5 \times 5) + (4 \times 6) + (3 \times 7) + (2 \times 8) + (1 \times 9) = 165.$$

Similarly, the number of ways the frog could make 8 clockwise and 3 anticlockwise jumps is 165. So the number of ways for the frog to make the 11 jumps is $165 + 165 = \mathbf{330}$.

Comment

Alternatively, for students who are familiar with binomial coefficients, the number of ways exactly 3 of the 11 jumps are clockwise is $\binom{11}{3} = 165$, and the number of ways exactly 8 of the 11 jumps are clockwise is $\binom{11}{8} = 165$.

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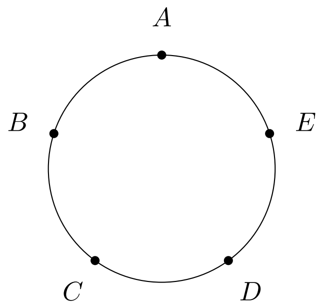
Method 2

Label the lily pads A, B, C, D, E in rotational order as shown. Let a_n, b_n, c_n, d_n, e_n be the number of ways the frog can take n jumps from A to A, B, C, D, E, respectively. We may assume the frog starts at pad A. So the aim is to find a_{11}.



As the frog may only jump from a given lily pad to an adjacent lily pad,

$$\begin{aligned} a_n &= e_{n-1} + b_{n-1} \\ b_n &= a_{n-1} + c_{n-1} \\ c_n &= b_{n-1} + d_{n-1} \\ d_n &= c_{n-1} + e_{n-1} \\ e_n &= d_{n-1} + a_{n-1}. \end{aligned}$$

As the frog starts at A, by symmetry we have that $b_n = e_n$ and $c_n = d_n$. Therefore we need only consider just three recursive equations $a_n = 2b_{n-1}, b_n = a_{n-1} + c_{n-1}$ and $c_n = b_{n-1} + c_{n-1}$, together with the initial condition $(a_1, b_1, c_1) = (0, 1, 0)$.

From these we generate this table:

n	1	2	3	4	5	6	7	8	9	10	11
a_n	0	2	0	6	2	20	14	70	72	254	330
b_n	1	0	3	1	10	7	35	36	127	167	474
c_n	0	1	1	4	5	15	22	57	93	220	385

Thus $a_{11} = \mathbf{330}$.

Method 3

Label the lily pads A, B, C, D, E in rotational order. Represent each of the lily pads with a column of 12 circles as shown. Draw a line connecting a circle in a row to a circle in the next row if the frog can jump from one corresponding lily pad to the next in one step. We may assume the frog starts at C. A number k in a circle in row n is the number of ways for the frog to reach the corresponding lily pad from C in n jumps.

All circles in row 0 have the number 0, the circles in row 1 have the numbers $0, 1, 0, 1, 0$ respectively, and the number in a circle in any other row is the sum of the numbers in the two adjacent circles in the previous row. We want the number in circle C in row 11.

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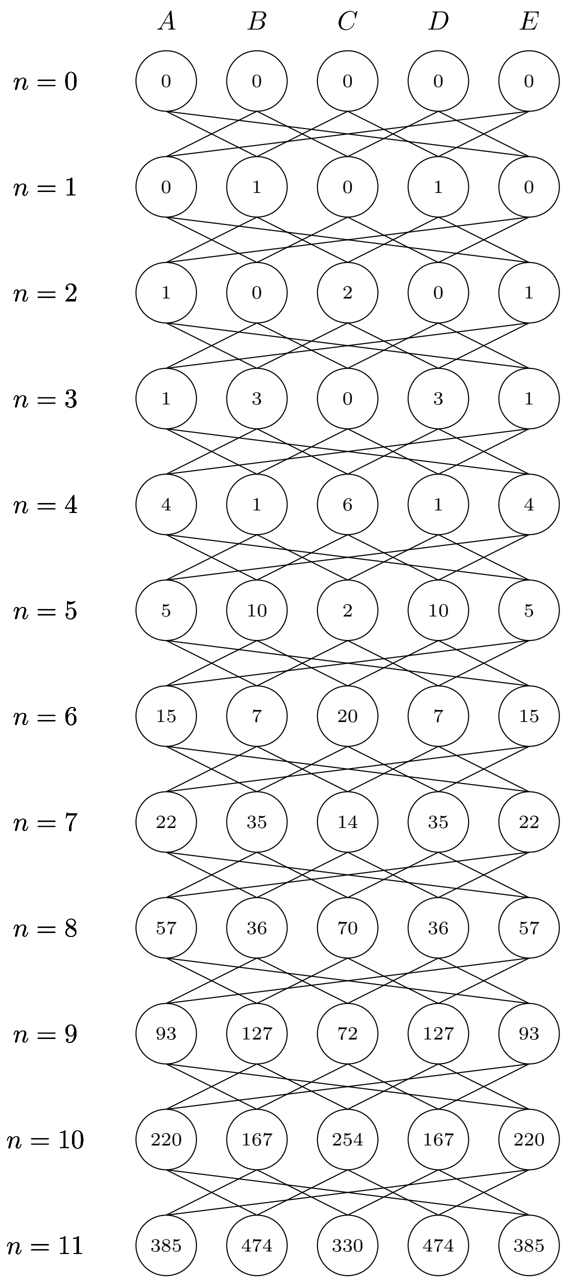

Thus the number of ways for the frog to take 11 jumps from C to C is **330**.

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6. *Method 1*

From (2) and (1) we have

$$\begin{aligned}(a+b)(c+1) &= (a+c)(b+1)\\ ac+a+bc+b &= ab+a+bc+c\\ a(c-b)+(b-c) &= 0\\ (a-1)(c-b) &= 0\end{aligned}$$

From (3) and (1) we have

$$\begin{aligned}(a+b)(c+1) &= (b+c)(a+1)\\ ac+a+bc+b &= ab+b+ac+c\\ a(b-1)+c(1-b) &= 0\\ (b-1)(a-c) &= 0\end{aligned}$$

So $a=1$ or $b=c$, and $b=1$ or $a=c$.

If $a=1$ and $b=1$, then the original equations give $2(c+1)=22$, hence $c=10$.

If $a=1$ and $a=c$, then the original equations give $2(b+1)=22$, hence $b=10$.

If $b=c$ and $b=1$, then the original equations give $2(a+1)=22$, hence $a=10$.

If $b=c$ and $a=c$, then the original equations give $2a(a+1)=22$, hence $a=-\frac{1}{2}\pm\frac{3}{2}\sqrt{5}$.

Thus there are only 5 solutions for (a,b,c):

$$(1,1,10),\quad (1,10,1),\quad (10,1,1),\quad (-\tfrac{1}{2}\pm\tfrac{3}{2}\sqrt{5},-\tfrac{1}{2}\pm\tfrac{3}{2}\sqrt{5},-\tfrac{1}{2}\pm\tfrac{3}{2}\sqrt{5}).$$

So the sum of all values of a is $1+1+10+(-\frac{1}{2}+\frac{3}{2}\sqrt{5})+(-\frac{1}{2}-\frac{3}{2}\sqrt{5})=\mathbf{11}$.

Method 2

From (2) and (1) we have

$$\begin{aligned}(a+b)(c+1) &= (a+c)(b+1)\\ ac+a+bc+b &= ab+a+bc+c\\ a(c-b)+(b-c) &= 0\\ (a-1)(b-c) &= 0 \qquad (4)\end{aligned}$$

Since equations (1), (2), (3) are symmetric in a,b,c, we also have:

$$\begin{aligned}(b-1)(a-c) &= 0 \qquad (5)\\ (c-1)(a-b) &= 0 \qquad (6)\end{aligned}$$

If no two of a,b,c are equal, then $a=1$, $b=1$, and $c=1$, a contradiction.

If $a=b\neq c$, then equation (4) gives $a=1$. Then equation (1) gives $2(c+1)=22$, hence $c=10$. So $(a,b,c)=(1,1,10)$.

By symmetry, if $a=c\neq b$, then $(a,b,c)=(1,10,1)$, and if $b=c\neq a$, then $(a,b,c)=(10,1,1)$.

If $a=b=c$, then equation (1) gives $a(a+1)=11$, hence $a=-\frac{1}{2}\pm\frac{3}{2}\sqrt{5}$.

Thus there are only 5 solutions for (a,b,c):

$$(1,1,10),\quad (1,10,1),\quad (10,1,1),\quad (-\tfrac{1}{2}\pm\tfrac{3}{2}\sqrt{5},-\tfrac{1}{2}\pm\tfrac{3}{2}\sqrt{5},-\tfrac{1}{2}\pm\tfrac{3}{2}\sqrt{5}).$$

So the sum of all values of a is $1+1+10+(-\frac{1}{2}+\frac{3}{2}\sqrt{5})+(-\frac{1}{2}-\frac{3}{2}\sqrt{5})=\mathbf{11}$.

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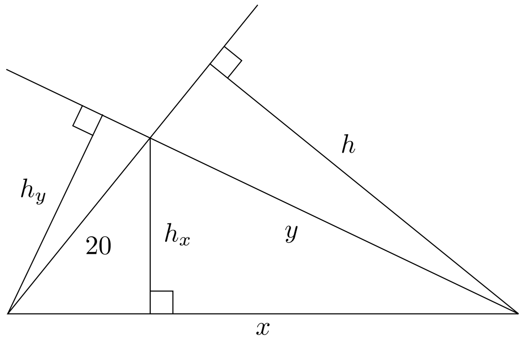
7. The area of the triangle is $\frac{1}{2} \times 20 \times h = \frac{1}{2} \times x \times h_x = \frac{1}{2} \times y \times h_y$.



Since $h = h_x + h_y$, we have $\frac{1}{20} = \frac{1}{x} + \frac{1}{y}$.

Method 1

Hence $y = \frac{20x}{x-20} = \frac{400}{x-20} + 20$. Therefore, $x - 20$ is a factor of 400.

A triangle inequality gives $20 + \frac{20x}{x-20} > x$, hence $x^2 - 60x + 400 < 0$ and $x < 10(\sqrt{5} + 3) < 53$.

A triangle inequality gives $x + 20 > \frac{20x}{x-20}$, hence $x^2 - 20x - 400 > 0$ and $x > 10(\sqrt{5} + 1) > 32$.

So $12 < x - 20 < 33$. Hence $x - 20 = 16, 20$, or 25.

Therefore $x = 36, 40$, or 45, with $y = 45, 40$, or 36 respectively.

Since $x > y$, we have $x = 45$, $y = 36$ and the perimeter is $20 + 36 + 45 = \mathbf{101}$.

Method 2

From $\frac{1}{20} = \frac{1}{x} + \frac{1}{y}$ we have $(x - 20)(y - 20) = 400$.
Thus $x - 20$ and $y - 20$ are cofactors of 400.

Since $x > y$, $x - 20 > \sqrt{400} = 20$ and $y - 20 < 20$.
A triangle inequality gives $x < 20 + y$.
Thus $20 < x - 20 < y < 40$.

So $x - 20 = 25$ and $y - 20 = 16$.
Hence $x = 45$, $y = 36$, and the perimeter is $45 + 36 + 20 = \mathbf{101}$.

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8. *Method 1*

A word that has its letters in alphabetical order and no repeated letter is called a *standard word*. Applying operations (a) and (b) shows that any word relates to a standard word. We claim that two words are related if and only if they relate to the same standard word.

First suppose that words X and Y relate to the same standard word S. Let Q denote a sequence of operations (a) and (b) that converts Y to S. Let Q' be the sequence Q with each operation (b) replaced with (c). Then the sequence of operations that convert X to S followed by Q', converts X to Y. Thus X and Y are related.

Conversely, suppose two words X and Y are related. None of the operations (a), (b), (c) change the parity of the number of times a letter L appears from one word to the next. So L appears an odd number of times in X if and only if it appears an odd number of times in Y. Also, a standard form that relates to X has exactly those letters in X that appear an odd number of times in X. The same applies to Y. Hence X and Y relate to the same standard form.

So the claim is true. In particular no two standard words are related. Hence the maximum number of unrelated words in a set is the number of standard words. We count the number of standard words by considering each of the nine letters in turn: it may be either included or excluded. So the maximum number of unrelated words is $2^9 = \mathbf{512}$.

Method 2

We show that two words X and Y are related if and only if, for each letter L, the parity of the number of times L appears in X is the same as the parity of the number of times L appears in Y.

First suppose that X and Y are related. None of the operations (a), (b), (c) change the parity of the number of times L appears from one word to the next. So the parities of L agree.

Conversely, suppose the parities agree. Operation (b) decreases the number of times a letter appears by 2. Operation (c) increases the number of times a letter appears by 2. So operations (a), (b), (c) can change X to a word that has each letter appearing the same number of times that it does in Y, and then operation (a) can arrange the letters so they are in the same order as in Y. So X and Y are related.

Define the *signature* of a word to be the binary sequence of 0s and 1s whose kth term has the same parity as the parity of the number of times the kth letter in the sequence A, B, C, D, E, F, G, H, I appears in the word. So two words relate if and only if they have the same signature. Hence the maximum number of unrelated words in a set is the number of signatures. We count the number of signatures by considering each of the terms in turn: it may be either 0 or 1. So the maximum number of unrelated words is $2^9 = \mathbf{512}$.

9. *Method 1*

First observe that, modulo 100, we have:

$9^1 = 9,\ 9^2 = 81,\ 9^3 \equiv 29,\ 9^4 \equiv 61,\ 9^5 \equiv 49,\ 9^6 \equiv 41,\ 9^7 \equiv 69,\ 9^8 \equiv 21,\ 9^9 \equiv 89,\ 9^{10} \equiv 1$

and

$91^1 = 91,\ 91^2 = 81,\ 91^3 \equiv 71,\ 91^4 \equiv 61,\ 91^5 \equiv 51,\ 91^6 \equiv 41,\ 91^7 \equiv 31,\ 91^8 \equiv 21,\ 91^9 \equiv 11,\ 91^{10} \equiv 1.$

Hence, for all integers $k \geq 0$, we have:

$9^{10k} \equiv 1,\ 9^{10k+1} = 9,\ 9^{10k+2} = 81,\ 9^{10k+3} \equiv 29,\ 9^{10k+4} \equiv 61,$
$9^{10k+5} \equiv 49,\ 9^{10k+6} \equiv 41,\ 9^{10k+7} \equiv 69,\ 9^{10k+8} \equiv 21,\ 9^{10k+9} \equiv 89$

and

$91^{10k} \equiv 1,\ 91^{10k+1} = 91,\ 91^{10k+2} = 81,\ 91^{10k+3} \equiv 71,\ 91^{10k+4} \equiv 61,$
$91^{10k+5} \equiv 51,\ 91^{10k+6} \equiv 41,\ 91^{10k+7} \equiv 31,\ 91^{10k+8} \equiv 21,\ 91^{10k+9} \equiv 11.$

Thus, if n is even, then 9^n and 91^n are both congruent to 1 modulo 10, hence
$9^{9^n} + 91^{91^n} \equiv 9 + 91 \bmod 100 \equiv 0 \bmod 100.$

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If n is odd, then 9^n is congruent to 9 modulo 10 and 91^n is congruent to 1 modulo 10, hence $9^{9^n} + 91^{91^n} \equiv 89 + 91 \bmod 100 = 80 \bmod 100$.

Thus $9^{9^n} + 91^{91^n}$ is divisible by 100 if and only if n is even.

Method 2

We work modulo 100 throughout the following.

First observe that:

$9^1 = 9,\ 9^2 = 81,\ 9^3 \equiv 29,\ 9^4 \equiv 61,\ 9^5 \equiv 49,\ 9^6 \equiv 41,\ 9^7 \equiv 69,\ 9^8 \equiv 21,\ 9^9 \equiv 89,\ 9^{10} \equiv 1$
and
$91^1 = 91,\ 91^2 = 81,\ 91^3 \equiv 71,\ 91^4 \equiv 61,\ 91^5 \equiv 51,\ 91^6 \equiv 41,\ 91^7 \equiv 31,\ 91^8 \equiv 21,\ 91^9 \equiv 11,\ 91^{10} \equiv 1.$

$9^{9^n} = (9^{9^2})^{9^{n-2}} = (9^{81})^{9^{n-2}} = (9^{80} \times 9)^{9^{n-2}} \equiv (1 \times 9)^{9^{n-2}} = 9^{9^{n-2}}.$

So $9^{9^n} \equiv \begin{cases} 9^{9^0} = 9 & \text{if } n \text{ is even} \\ 9^{9^1} \equiv 89 & \text{if } n \text{ is odd} \end{cases}$

$91^{91^n} = (91^{91^2})^{91^{n-2}} = (91^{8281})^{91^{n-2}} = (91^{8280} \times 91)^{91^{n-2}} \equiv (1 \times 91)^{91^{n-2}} = 91^{91^{n-2}}.$

So $91^{91^n} \equiv \begin{cases} 91^{91^0} = 91 & \text{if } n \text{ is even} \\ 91^{91^1} \equiv 91 & \text{if } n \text{ is odd} \end{cases}$

Hence $9^{9^n} + 91^{91^n} \equiv \begin{cases} 9 + 91 = 100 & \text{if } n \text{ is even} \\ 89 + 91 \equiv 80 & \text{if } n \text{ is odd} \end{cases}$

Thus $9^{9^n} + 91^{91^n}$ is divisible by 100 if and only if n is even.

Method 3

We work modulo 100 throughout the following.

First observe that:

$9^1 = 9,\ 9^2 = 81,\ 9^3 \equiv 29,\ 9^4 \equiv 61,\ 9^5 \equiv 49,\ 9^6 \equiv 41,\ 9^7 \equiv 69,\ 9^8 \equiv 21,\ 9^9 \equiv 89,\ 9^{10} \equiv 1$
and
$91^1 = 91,\ 91^2 = 81,\ 91^3 \equiv 71,\ 91^4 \equiv 61,\ 91^5 \equiv 51,\ 91^6 \equiv 41,\ 91^7 \equiv 31,\ 91^8 \equiv 21,\ 91^9 \equiv 11,\ 91^{10} \equiv 1.$

$$\begin{aligned} 9^{9^n} + 91^{91^n} &\equiv 9^{9^n} + (-9)^{91^n} \\ &= 9^{9^n} + (-1)^{91^n} 9^{91^n} \\ &= 9^{9^n} - 9^{91^n} \\ &= 9^{9^n}(1 - 9^{91^n - 9^n}) \end{aligned}$$

Now $9^{100q+r} = (9^{100})^q \times 9^r \equiv 1^q \times 9^r = 9^r$.

If n is even, then $91^n - 9^n \equiv 0$, hence $9^{91^n - 9^n} \equiv 9^0$.
Therefore $9^{9^n} + 91^{91^n} \equiv 9^{9^n}(1-1) = 0$.

If n is odd, then $91^n - 9^n$ is congruent to one of
$91 - 9 = 82, 71 - 29 = 42, 51 - 49 = 2, 31 - 69 = -38 \equiv 62, 11 - 89 = -78 \equiv 22.$
Hence $9^{91^n - 9^n}$ is congruent to one of 9^2, 9^{22}, 9^{42}, 9^{62}, 9^{82}, all of which are congruent to $9^2 = 81$.
So $9^{9^n} + 91^{91^n} \equiv 9^{9^n}(1 - 81) \equiv 20 \times 9^{9^n}$.
Now 9^n is equivalent to one of 9^9, 9^{29}, 9^{49}, 9^{69}, 9^{89}, all of which are congruent to $9^9 \equiv 89$.
Hence $9^{9^n} + 91^{91^n} \equiv 20 \times 89 = 1780 \equiv 80$.

Thus $9^{9^n} + 91^{91^n}$ is divisible by 100 if and only if n is even.

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Method 4

We first show that $9^{9^n} + 91^{91^n}$ is divisible by 4 for all positive integers n.

Modulo 4, $9^k \equiv 1$ and $91^k \equiv 3$ for all odd k.
Since 9^n and 91^n are odd for all n, $9^{9^n} + 91^{91^n} \equiv 1 + 3 \equiv 0$ for all n.

Next we show that $9^{9^n} + 91^{91^n}$ is divisible by 25 if and only if n is even.

Modulo 25, we have:
$9^1 = 9,\ 9^2 \equiv 6,\ 9^3 \equiv 4,\ 9^4 \equiv 11,\ 9^5 \equiv 24,\ 9^6 \equiv 16,\ 9^7 \equiv 19,\ 9^8 \equiv 21,\ 9^9 \equiv 14,\ 9^{10} \equiv 1$
and
$91^1 = 16,\ 91^2 = 6,\ 91^3 \equiv 21,\ 91^4 \equiv 11,\ 91^5 \equiv 1.$

Hence $9^{20} \equiv 1 \equiv 91^{20}$ mod 25.

Modulo 20, we have $9^1 = 9,\ 9^2 \equiv 1$, and $91^1 = 11,\ 91^2 \equiv 1$.

Hence, for even n, we have:
$9^{9^n} = 9^{20r+1} = (9^{20})^r \times 9 \equiv 9$ mod 25 and $91^{91^n} = 91^{20r+1} = (91^{20})^r \times 91 \equiv 91 \equiv 16$ mod 25.
Hence $9^{9^n} + 91^{91^n} \equiv 9 + 16 \equiv 0$ mod 25.

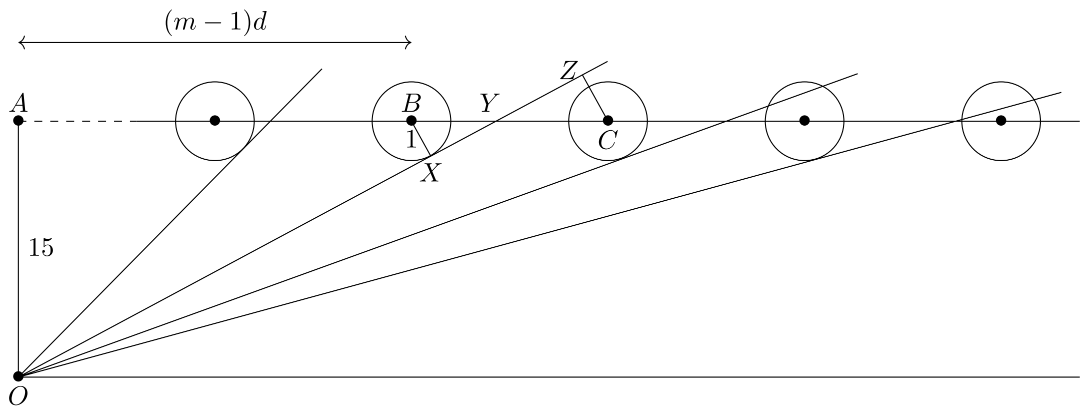
For odd n, we have:
$9^{9^n} = 9^{20r+9} = (9^{20})^r \times 9^9 \equiv 14$ mod 25 and $91^{91^n} = 91^{20r+11} = (91^{20})^r \times 91^{11} \equiv 16$ mod 25.
Hence $9^{9^n} + 91^{91^n} \equiv 14 + 16 \equiv 5$ mod 25.

Thus $9^{9^n} + 91^{91^n}$ is divisible by 100 if and only if n is even.

10. Suppose a tower with centre C is completely visible to the observer but that tower obscures part of the next tower away from O. By drawing lines through O tangential to the towers, we see that all towers from A to C are completely visible from O but none of the towers to the right of C is visible. Number the tower centres from A to C: 0, 1, 2, …, m. We first show that $m \leq 7$.

Method 1

Line OZ is tangential at X to the $(m-1)$th tower with centre B as shown. It intersects $\mathcal{L}$ at Y and CZ is perpendicular to OZ.



Triangles BXY and CYZ are similar, hence $BY \leq YC$. So $2BY \leq BC = d$.
Triangles BXY and OAY are similar, hence $OY/15 = BY/1$.

Since there is a tower centre at A, we have
$AY = AB + BY = (m-1)d + BY \geq (2m-1)BY = (2m-1)OY/15.$
Since $AY/OY < 1$, we have $15 > 2m - 1$, hence $m \leq 7$.

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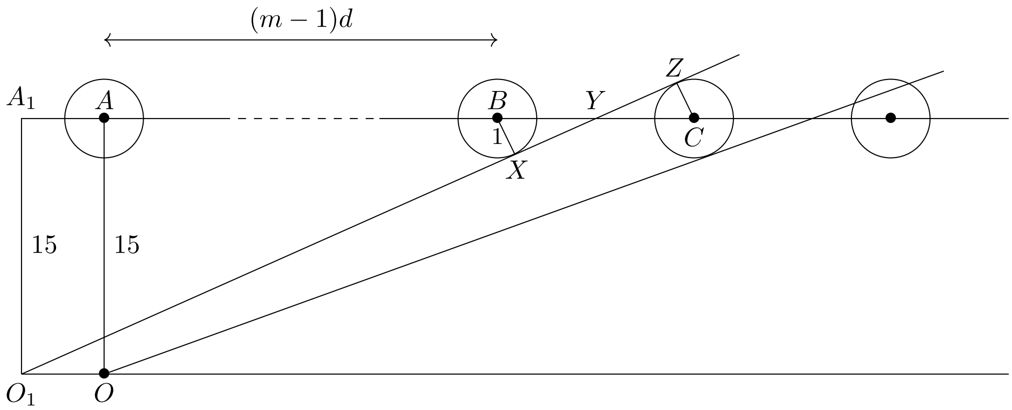
Method 2

Between the $(m-1)$th tower, with centre B, and the mth tower, with centre C, draw a line that is tangential to both towers at X and Z respectively as shown. Let O_1 be a point on that line 15 metres from $\mathcal{L}$. Let A_1 be the point on $\mathcal{L}$ closest to O_1. Note that O_1A_1 will be on the left of OA.

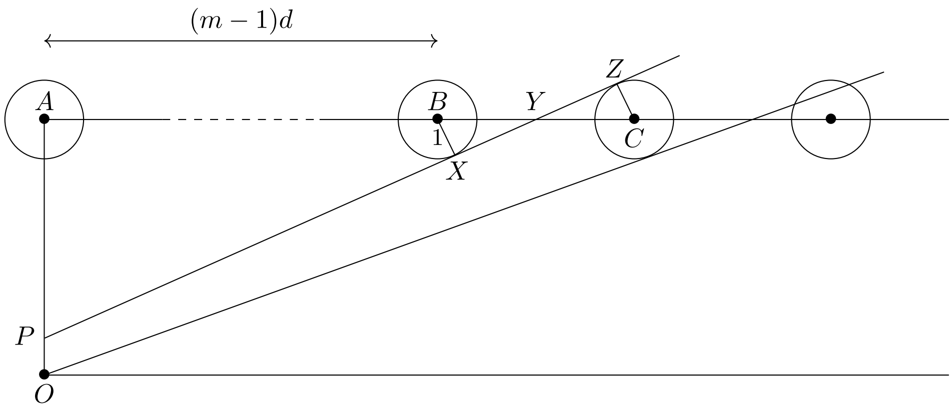


Triangles BXY and CYZ are similar, hence $BY = YC$. So $2BY = d$.
Triangles BXY and O_1A_1Y are similar, hence $A_1Y = 15XY$ and $O_1Y = 15BY$.

We have $A_1Y \geq AY = AB + BY = (m-1)d + d/2 = (m - \frac{1}{2})d$.
So $(m-\frac{1}{2})^2 d^2 \leq A_1Y^2 = 225XY^2 = 225(BY^2 - 1) = 225(\frac{1}{4}d^2 - 1) < 225d^2/4$.
Hence $m < 15/2 + 1/2 = 8$, so $m \leq 7$.

Method 3

Between the $(m-1)$th tower, with centre B, and the mth tower, with centre C, draw a line that is tangential to both towers at X and Z respectively as shown. Let that line intersect OA at P.



Triangles BXY and CYZ are similar, hence $BY = YC$. So $2BY = d$.
Triangles BXY and PAY are similar, hence $AP = AY/XY$.

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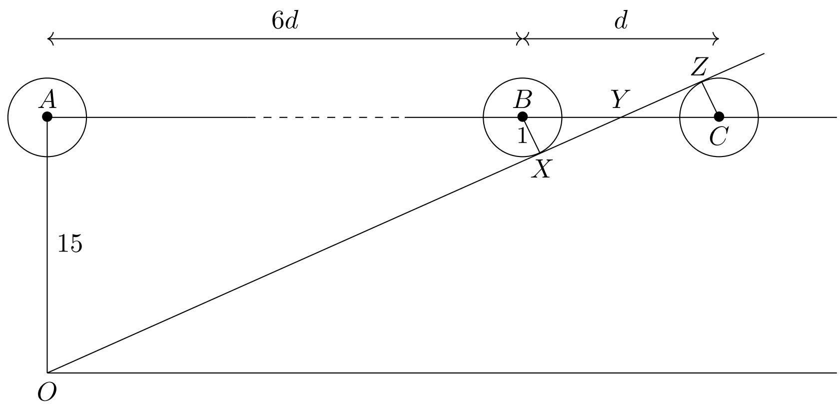
So we have

$$\begin{aligned} 15 \geq \frac{AY}{XY} &= \frac{(m-1)d + d/2}{\sqrt{d^2/4 - 1}} \\ 15\sqrt{d^2-4} &\geq 2(m-1)d + d = (2m-1)d \\ m &\leq (15\sqrt{1-4/d^2}+1)/2 \\ &= 15\sqrt{1/4 - 1/d^2} + 1/2 \\ &< 15/2 + 1/2 = 8 \end{aligned}$$

Hence $m \leq 7$.

Next we show that $m = 7$ is possible by finding an acceptable value for d that allows the following configuration.



As above, from similar triangles, $d = 2BY$ and $OY = 15BY$. We have

$$\begin{aligned} OY^2 &= OA^2 + AY^2 \\ 225(d/2)^2 &= 225 + (13d/2)^2 \\ 225d^2 &= 900 + 169d^2 \\ d^2 &= 900/56 > 4 \end{aligned}$$

Since $d > 2$, it is possible to have $m = 7$.

Thus the maximum number of towers on the right of A and completely visible at O is 7. By symmetry, the maximum number of towers on the left of A and completely visible at O is 7. Including the tower at A, the maximum number of towers completely visible at O is $2 \times 7 + 1 = 15$.

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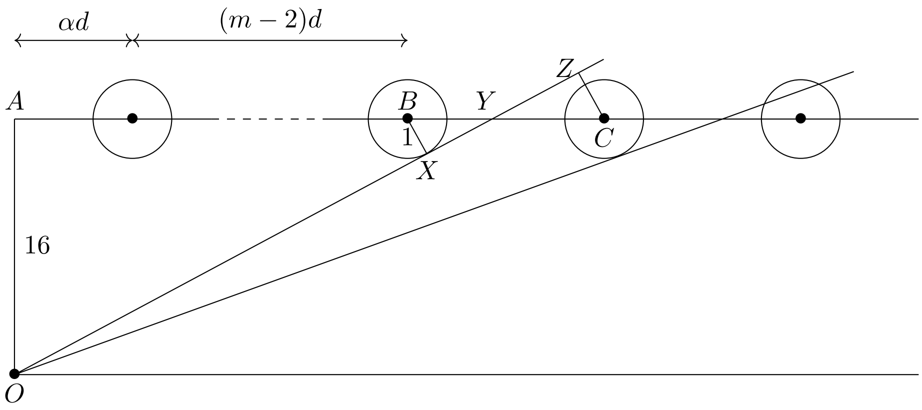
Investigation

If there is no tower centre at A, let the distance between A and the closest tower centre on the right of A be αd where $0 < \alpha < 1$. Suppose the observer can completely see exactly m towers to the right of A and m' towers to the left of A.

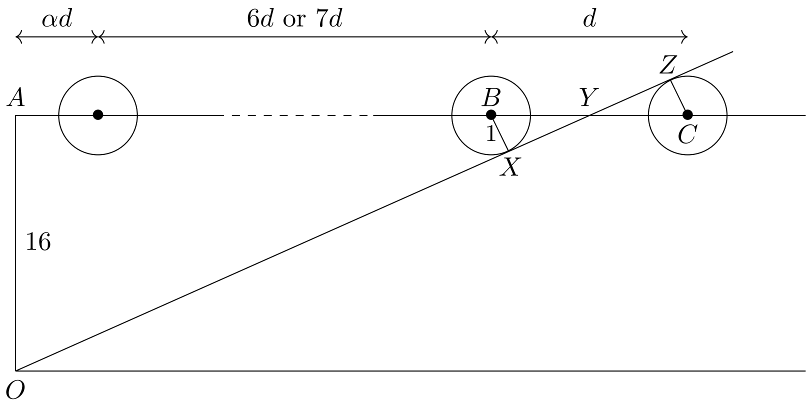


As above, from similar triangles, $d \geq 2BY$ and $OY = 16BY$.

Now $AY = \alpha d + (m-2)d + BY \geq (2\alpha + 2(m-2)+1)BY = (2\alpha + 2m - 3)OY/16$.
Since $AY/OY < 1$, we have $m < 9.5 - \alpha$. Similarly $m' < 9.5 - (1-\alpha) = 8.5 + \alpha$.

If $\alpha = 1/2$, then $m \leq 8$ and $m' \leq 8$.
If $\alpha < 1/2$, then $m \leq 9$ and $m' \leq 8$.
If $\alpha > 1/2$, then $m \leq 8$ and $m' \leq 9$.

Next we show that these upper bounds on m are attainable by finding an acceptable value for d that allows the following configurations.



As above, from similar triangles, $d = 2BY$ and $OY = 16BY$. We have

$$\begin{aligned} OY^2 &= OA^2 + AY^2 \\ 256(d/2)^2 &= 256 + d^2(\alpha + (6 \text{ or } 7) + \tfrac{1}{2})^2 \\ 256d^2 &= 1024 + d^2(2\alpha + (13 \text{ or } 15))^2 \end{aligned}$$

With $1 > \alpha \geq 1/2$, we have $256d^2 = 1024 + d^2(2\alpha+13)^2$. So $d^2 = 1024/(256 - (2\alpha+13)^2)$.

With $0 < \alpha < 1/2$, we have $256d^2 = 1024 + d^2(2\alpha+15)^2$. So $d^2 = 1024/(256 - (2\alpha+15)^2)$.

In each case there is a value for $d > 2$. So it is possible for m to attain its upper bounds. Similarly, the upper bounds for m' are attainable.

So the maximum number of completely visible towers is $8+8=16$ if $\alpha = 1/2$ and $8+9=17$ if $\alpha \neq 1/2$.

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Statistics

Distribution of Awards/School Year

Year	Number of Students	Number of Awards				
		Prize	High Distinction	Distinction	Credit	Participation
8	476	2	19	40	125	290
9	539	6	51	53	172	257
10	493	16	71	83	153	170
Other	455	5	11	28	92	319
All Years	**1963**	**29**	**152**	**204**	**542**	**1036**

The award distribution is based on approximately the top 10% for High Distinction, next 15% for Distinction and the following 25% for Credit.

Number of Correct Answers Questions 1–8

Year	Number Correct / Question							
	1	2	3	4	5	6	7	8
8	451	174	253	161	25	37	32	32
9	520	224	291	230	69	63	45	52
10	478	220	299	288	90	61	66	97
Other	430	109	202	110	29	41	21	26
All Years	**1879**	**727**	**1045**	**789**	**213**	**202**	**164**	**207**

Mean Score/Question/School Year

School Year	Number of Students	Question			Overall Mean
		1–8	9	10	
8	476	6.9	0.4	0.0	7.3
9	539	8.3	0.6	0.1	9.0
10	493	10.1	1.0	0.2	11.3
Other	455	6.0	0.3	0.0	6.4
All Years	**1963**	**7.9**	**0.6**	**0.1**	**8.6**