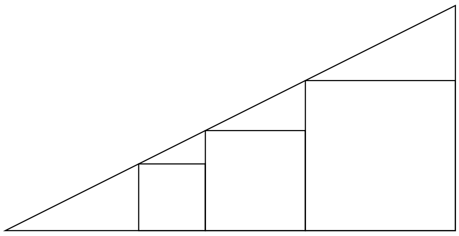
Problems

1. Positive integers a and b satisfy $(a+b)(a-b) = 2023$ and $\frac{a}{b} + \frac{b}{a} = \frac{25}{12}$. Find the value of a.
[**2 marks**]

2. Three squares lie inside a right-angled triangle as shown. The side lengths of the smallest and largest squares are 28 and 63 respectively. Find the side length of the middle square.



[**2 marks**]

3. The ten pairwise (two at a time) sums of five distinct integers are 0, 1, 2, 4, 7, 8, 9, 10, 11, 12. Find the sum of the five integers. [**2 marks**]

4. Given that there is only one pair of numbers $\{a, b\}$ such that $16a^ab^b = 81a^bb^a$, find $a+b$.
[**2 marks**]

5. There are several different ways of arranging the numbers 1, 2, 3, 4, 5, 6 in a line. Each of these arrangements can be the base of a pyramid in which each row is formed from the one below it by writing the sum of each pair of adjacent numbers. For example, the following pyramid is built on the arrangement 3, 6, 1, 5, 4, 2.

$$\begin{array}{ccccccccccc}
 & & & & & 115 & & & & & \\
 & & & & 57 & & 58 & & & & \\
 & & & 29 & & 28 & & 30 & & & \\
 & & 16 & & 13 & & 15 & & 15 & & \\
 & 9 & & 7 & & 6 & & 9 & & 6 & \\
3 & & 6 & & 1 & & 5 & & 4 & & 2
\end{array}$$

How many arrangements of the numbers 1 to 6 in a pyramid base produce a top number that is a multiple of 5? [**3 marks**]

6. Two local sports teams, the Tigers and the Lions, are coming together for some practice. There are 10 Tigers and 10 Lions. They are to be arranged into 10 Tiger–Lion pairs. To make the game as competitive as possible, we want to avoid height mismatches. So, each Tiger is assigned a number from 1 to 10 in ascending order of heights, and each Lion is assigned a number from 1 to 10 in ascending order of heights. A Tiger may be paired up with a Lion if and only if their numbers differ by no more than 1. For example, Tiger 4 may pair up with Lions 3, 4 or 5, but not 2 and not 6. How many ways can the Tigers and Lions be paired up? [**4 marks**]

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7. The number $1/137$, written as a decimal, is $0.00729927\,00729927\ldots$, which repeats every 8 digits after the decimal point (but no smaller number of digits repeats.) What is the smallest n such that $1/n$, when written as a decimal, repeats every 8 digits after the decimal point (but no smaller number of digits repeats)? [**5 marks**]

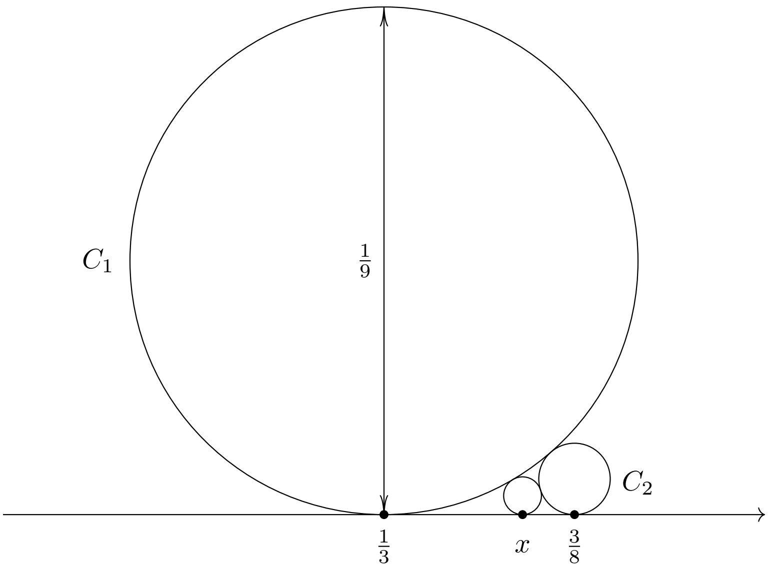
8. Each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 is coded by a letter selected from A to J with no two digits having the same letter. Find the 3-digit number coded by DEG if the integers corresponding to ABACDE, CAFDG and CHHBAED (with A, C $\neq 0$) are known to be the side lengths of a triangle. [**5 marks**]

9. Determine the number n of real solution pairs (x, y) of the simultaneous equations:

$$x^3 - y^3 = 7(x - y)$$
$$x^3 + y^3 = 5(x + y)$$

Find the sum of the corresponding (not necessarily distinct) n values of $x^2 + y^2$. [**5 marks**]

10. Two circles C_1, C_2 are placed tangent to the real number line and externally tangent to each other. Circle C_1 is tangent to the line at $1/3$ and has diameter $1/9$. Circle C_2 is tangent to the line at $3/8$. A third circle C_3 is then placed tangent to C_1, C_2, and to the real number line at $x < 3/8$. Find x and the radius of C_3.



[**5 marks**]

Investigation

Consider the sequence of circles $C_1, C_2, C_3, C_4, \ldots$, beginning with circles C_1 and C_2 above and their tangent points $x_1 = \frac{1}{3}$ and $x_2 = \frac{3}{8}$ on the real line, and continuing so that, for $n \geq 3$, C_n is the circle tangent to C_1, C_{n-1}, and to the real number line at a point $x_n < x_{n-1}$.
For $n \geq 4$, find expressions for x_n and the radius r_n of C_n. [**4 bonus marks**]

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