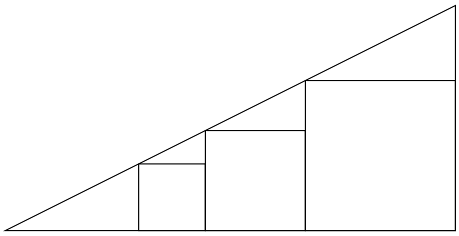
Problems

1. Positive integers a and b satisfy $(a+b)(a-b) = 2023$ and $\frac{a}{b} + \frac{b}{a} = \frac{25}{12}$. Find the value of a.
[**2 marks**]

2. Three squares lie inside a right-angled triangle as shown. The side lengths of the smallest and largest squares are 28 and 63 respectively. Find the side length of the middle square.



[**2 marks**]

3. The ten pairwise (two at a time) sums of five distinct integers are 0, 1, 2, 4, 7, 8, 9, 10, 11, 12. Find the sum of the five integers. [**2 marks**]

4. Given that there is only one pair of numbers $\{a, b\}$ such that $16a^a b^b = 81a^b b^a$, find $a + b$.
[**2 marks**]

5. There are several different ways of arranging the numbers 1, 2, 3, 4, 5, 6 in a line. Each of these arrangements can be the base of a pyramid in which each row is formed from the one below it by writing the sum of each pair of adjacent numbers. For example, the following pyramid is built on the arrangement 3, 6, 1, 5, 4, 2.

115

57 58

29 28 30

16 13 15 15

9 7 6 9 6

3 6 1 5 4 2

How many arrangements of the numbers 1 to 6 in a pyramid base produce a top number that is a multiple of 5? [**3 marks**]

6. Two local sports teams, the Tigers and the Lions, are coming together for some practice. There are 10 Tigers and 10 Lions. They are to be arranged into 10 Tiger–Lion pairs. To make the game as competitive as possible, we want to avoid height mismatches. So, each Tiger is assigned a number from 1 to 10 in ascending order of heights, and each Lion is assigned a number from 1 to 10 in ascending order of heights. A Tiger may be paired up with a Lion if and only if their numbers differ by no more than 1. For example, Tiger 4 may pair up with Lions 3, 4 or 5, but not 2 and not 6. How many ways can the Tigers and Lions be paired up? [**4 marks**]

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7. The number $1/137$, written as a decimal, is $0.00729927\,00729927\ldots$, which repeats every 8 digits after the decimal point (but no smaller number of digits repeats.) What is the smallest n such that $1/n$, when written as a decimal, repeats every 8 digits after the decimal point (but no smaller number of digits repeats)? [**5 marks**]

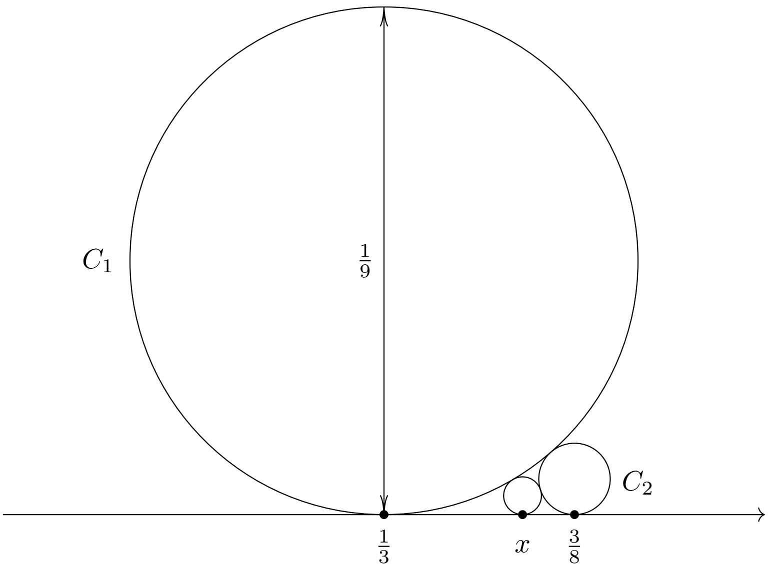
8. Each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 is coded by a letter selected from A to J with no two digits having the same letter. Find the 3-digit number coded by DEG if the integers corresponding to ABACDE, CAFDG and CHHBAED (with A, C $\neq 0$) are known to be the side lengths of a triangle. [**5 marks**]

9. Determine the number n of real solution pairs (x, y) of the simultaneous equations:

$$x^3 - y^3 = 7(x - y)$$
$$x^3 + y^3 = 5(x + y)$$

Find the sum of the corresponding (not necessarily distinct) n values of $x^2 + y^2$. [**5 marks**]

10. Two circles C_1, C_2 are placed tangent to the real number line and externally tangent to each other. Circle C_1 is tangent to the line at $1/3$ and has diameter $1/9$. Circle C_2 is tangent to the line at $3/8$. A third circle C_3 is then placed tangent to C_1, C_2, and to the real number line at $x < 3/8$. Find x and the radius of C_3.



[**5 marks**]

Investigation

Consider the sequence of circles $C_1, C_2, C_3, C_4, \ldots$, beginning with circles C_1 and C_2 above and their tangent points $x_1 = \frac{1}{3}$ and $x_2 = \frac{3}{8}$ on the real line, and continuing so that, for $n \geq 3$, C_n is the circle tangent to C_1, C_{n-1}, and to the real number line at a point $x_n < x_{n-1}$.
For $n \geq 4$, find expressions for x_n and the radius r_n of C_n. [**4 bonus marks**]

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Solutions

1. *Method 1*

Since $2023 = 7 \times 17^2$, we have from the first equation $(a+b)(a-b) = 1 \times 2023 = 7 \times 289 = 17 \times 119$.
Since $a+b > a-b$, we have the following three pairs of simultaneous equations.

- $a+b = 2023$ and $a-b = 1$. Adding gives $a = 1012$, hence $b = 1011$, then $\frac{a}{b} + \frac{b}{a} < 1 + \frac{1}{1011} + 1 < 2.001 < \frac{25}{12}$.
- $a+b = 289$ and $a-b = 7$. Adding gives $a = 148$, hence $b = 141$, then $\frac{a}{b} + \frac{b}{a} < 1 + \frac{7}{141} + 1 < 2.05 < \frac{25}{12}$.
- $a+b = 119$ and $a-b = 17$. Adding gives $a = 68$, hence $b = 51$.

Only the last of these (a,b) pairs satisfy the second given equation, so $a = \mathbf{68}$.

Method 2

From the second equation we have

$$\begin{aligned} 0 &= 12a^2 + 12b^2 - 25ab \\ &= (3a-4b)(4a-3b) \end{aligned}$$

Since a, b, and 2023 are positive, the first equation requires $b < a$. So $b = 3a/4$. Substituting in the first equation gives $(7a/4)(a/4) = 2023$, hence $a^2 = 289 \times 16$. So $a = 17 \times 4 = \mathbf{68}$.

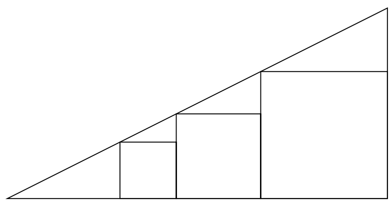
Method 3

From the first equation we have $a - b > 0$ and $(a+b)(a-b) = 7 \times 17^2$. From the second equation we have $a^2 + b^2 = 25ab/12$. This gives

$$\begin{aligned} (a+b)^2 &= a^2 + b^2 + 2ab = 49ab/12 \\ (a-b)^2 &= a^2 + b^2 - 2ab = ab/12 \\ 7ab/12 &= (a+b)(a-b) = 7 \times 17^2 \\ a - b &= 17 \\ a + b &= 7 \times 17 \\ 2a &= 8 \times 17 \\ a &= 4 \times 17 = \mathbf{68} \end{aligned}$$

2. *Method 1*

Let m be the side length of the middle square.



The triangles above the squares are similar. So we have

$$\begin{aligned} \frac{m-28}{28} &= \frac{63-m}{m} \\ \frac{m}{28} - 1 &= \frac{63}{m} - 1 \\ m^2 &= 28 \times 63 = 2^2 \cdot 7 \times 7 \cdot 3^2 \\ m &= 2 \cdot 3 \cdot 7 = \mathbf{42} \end{aligned}$$

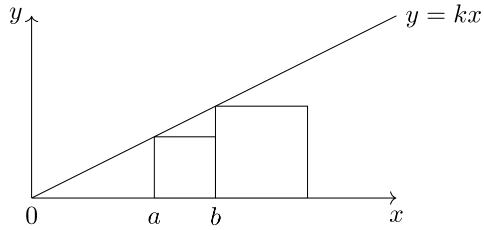
Method 2

Other similar triangles could be used, for example:

$$\frac{m-28}{28} = \frac{63-28}{m+28} = \frac{35}{m+28}$$
$$28 \times 35 = (m-28)(m+28) = m^2 - 28^2$$
$$m^2 = 28(35+28) = 28 \times 63 = 2^2 \cdot 7 \times 7 \cdot 3^2$$
$$m = 2 \cdot 3 \cdot 7 = \mathbf{42}$$

Method 3

Consider any two consecutive squares on the cartesian plane as shown with k some positive constant.



The ratio of the side length of the larger square to the side length of the smaller square is

$$\frac{kb}{ka} = \frac{b}{a} = \frac{a+ka}{a} = 1+k.$$

Thus the side lengths of the original three squares are in geometric progression.
So, if the side length of the original middle square is m, then $m/28 = 63/m$,
$m^2 = 28 \times 63 = 2^2 \cdot 7 \times 7 \cdot 3^2$, $m = 2 \cdot 3 \cdot 7 = \mathbf{42}$.

Comment

In general, if the two outer squares have side lengths a and c, then the middle square has side length $b = \sqrt{ac}$, that is, the geometric mean of a and c.

Method 1

Let the five integers be a, b, c, d, e, with $a < b < c < d < e$.
Then $a + b < a + c$ and all the other pair sums are larger. So $a + b = 0$, $a + c = 1$, $c = b + 1$.
Since $c + e < d + e$ and all the other pair sums are smaller, $d + e = 12$, $c + e = 11$, $d = c + 1$.

Thus b, c, d are consecutive integers. Therefore $a + d = 2$ and $b + e = 10$.

Since b, c, d are consecutive integers, so are $b + c, b + d$ and $c + d$. The only consecutive integers left are 7, 8, 9, so $b + c = 7, b + d = 8$, $c + d = 9$. The only pair sum left is $a + e = 4$.

Adding $a+b = 0$ and $a+c = 1$ gives $2a+b+c = 1$. But $b+c = 7$, so $2a = -6$ and $a = -3$. Then $b = 3, c = 4, d = 5$, $e = 7$. Therefore $a + b + c + d + e = -3 + 3 + 4 + 5 + 7 = \mathbf{16}$.

Method 2

Let the five integers be a, b, c, d, e, with $a < b < c < d < e$.
Then $a + b < a + c$ and all the other pair sums are larger. So $a + b = 0$, $a + c = 1$, $c = b + 1$.
Since $c + e < d + e$ and all the other pair sums are smaller, $d + e = 12$, $c + e = 11$, $d = c + 1$.

So $b + c = 2b + 1$ and $c + d = 2c + 1$, which are both odd, so must be 7 and 9 respectively since these are the only odd pair sums not accounted for. So $c = 4$ and $a + b + c + d + e = 0 + 4 + 12 = \mathbf{16}$.

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Method 3

We have $0+1+2+4+7+8+9+10+11+12=64$. Let the five integers be a,b,c,d,e. Then

$$\begin{aligned}64 &= (a+b)+(a+c)+(a+d)+(a+e)\\ &\qquad +(b+c)+(b+d)+(b+e)\\ &\qquad\qquad +(c+d)+(c+e)\\ &\qquad\qquad\qquad +(d+e)\\ &= 4(a+b+c+d+e).\end{aligned}$$

Hence $a+b+c+d+e=64/4=\mathbf{16}$.

Comment

Method 3 shows that we do not need to know that the integers are distinct.

4. The given equation is symmetric in a and b and clearly $a \neq b$, so we may assume $b > a$. We have

$$\begin{aligned}\frac{16}{81} &= \frac{a^b b^a}{a^a b^b}\\ \left(\frac{2}{3}\right)^4 &= \frac{a^{b-a}}{b^{b-a}} = \left(\frac{a}{b}\right)^{b-a}\end{aligned}$$

This suggests $\frac{2}{3}=\frac{a}{b}$ and $b-a=4$, from which we get $a=8$ and $b=12$.
Given that there is only one solution pair, we have $a+b=\mathbf{20}$.

Comment

The question states that the solution is unique for integers. Here is a proof.

We may assume $b > a$. Let d be the greatest common divisor of a and b.
Let $a=dx$ and $b=dy$. Then $y>x$ and the gcd of x and y is 1.

Substitution in the given equation gives $16y^{d(y-x)}=81x^{d(y-x)}$.
So $x^{d(y-x)}=16$, hence x is 2, 4, or 16.

If $x=2$, then $d(y-x)=4$, $y^4=81$, $y=3$, $d=4$, $a=8$ and $b=12$.
If $x=4$, then $d(y-x)=2$, $y^2=81$, $y=9$, and d is not an integer.
If $x=16$, then $d(y-x)=1$, $y=81$, and d is not an integer.

Thus $\{8,\ 12\}$ is the only solution pair for integers.

5. *Method 1*

The problem can be approached algebraically with a,b,c,d,e,f as the base arrangement.
Then the next row up is $a+b, b+c, c+d, d+e, e+f$.
The row above that is $a+2b+c, b+2c+d, c+2d+e, d+2e+f$.
The next row up is $a+3b+3c+d, b+3c+3d+e, c+3d+3e+f$.
The row above that is $a+4b+6c+4d+e, b+4c+6d+4e+f$.
Hence the top number is $a+5b+10c+10d+5e+f=5(b+2c+2d+e)+a+f$.
This is a multiple of 5 if and only if $a+f$ is a multiple of 5.

There are 6 possible pairs for $(a,\ f)$: $(1,\ 4)$, $(2,\ 3)$, $(3,\ 2)$, $(4,\ 1)$, $(4,\ 6)$, $(6,\ 4)$.
In each case, there are $4!$ ways of ordering the remaining 4 digits.

So the total number of arrangements of numbers 1 to 6 that produce a top number that is a multiple of 5 is $6 \times 24 = \mathbf{144}$.

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Method 2

The numbers in the following pyramid represent the number of times the corresponding number in the original pyramid contributes to the original top number.

$$\begin{array}{ccccccccccc}
&&&&&1&&&&& \\
&&&&1&&1&&&& \\
&&&1&&2&&1&&& \\
&&1&&3&&3&&1&& \\
&1&&4&&6&&4&&1& \\
1&&5&&10&&10&&5&&1
\end{array}$$

So, if the original bottom row is a, b, c, d, e, f, the top number is $a+5b+10c+10d+5e+f = 5(b+2c+2d+e)+a+f$. This is a multiple of 5 if and only if $a+f$ is a multiple of 5.

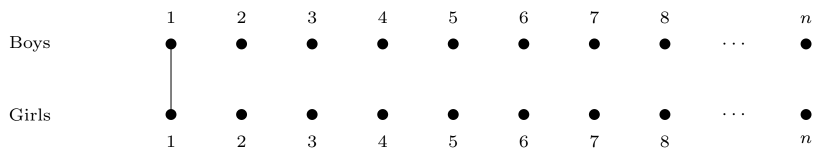
There are 6 possible pairs for $(a,\ f)$: (1, 4), (2, 3), (3, 2), (4, 1), (4, 6), (6, 4).
In each case, there are 4! ways of ordering the remaining 4 digits.

So the total number of arrangements of numbers 1 to 6 that produce a top number that is a multiple of 5 is $6 \times 24 = \mathbf{144}$.

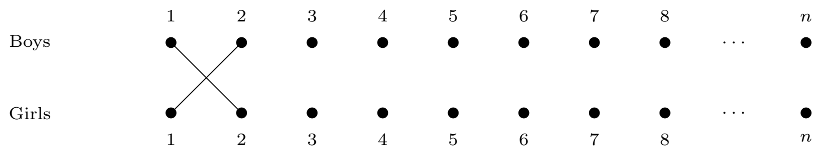
6. *Method 1*

It helps to proceed more generally. Let p_n be the number of permissible partnerings of n boys with n girls. Consider Boy 1. We can partner him with Girl 1 or Girl 2.

If we partner Boy 1 with Girl 1, then there are now $n-1$ boys and girls to partner according to the same rules. There are p_{n-1} permissible partnerings of the remaining boys and girls.



If we partner Boy 1 with Girl 2, then Girl 1 is forced to partner with Boy 2. There are now $n-2$ boys and girls to partner according to the same rules. There are p_{n-2} permissible partnerings of the remaining boys and girls.



Therefore, we find that $p_n = p_{n-1} + p_{n-2}$. Clearly $p_1 = 1$ and $p_2 = 2$. Therefore we can calculate p_{10} recursively.

n	1	2	3	4	5	6	7	8	9	10
p_n	1	2	3	5	8	13	21	34	55	89

So the number of ways the 10 boys and girls can be partnered is **89**.

Comment

Curiously, p_n is the Fibonacci sequence.

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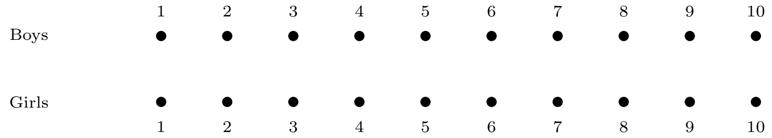
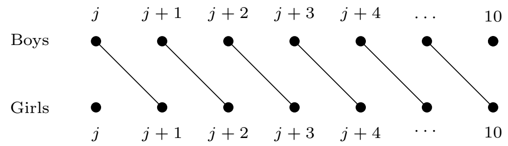
Method 2

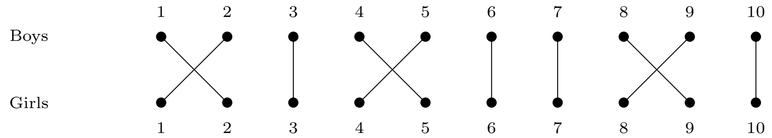
Each boy and girl is represented by a node labelled from 1 to 10.



Suppose Boy j partners with Girl $j+1$. Then Boy $j+1$ must partner with Girl j or Girl $j+2$. However, if he partners with Girl $j+2$, then Boy $j+2$ is forced to partner with Girl $j+3$, and Boy $j+3$ is forced to partner with Girl $j+4$, and so on. This leaves the final boy with no partner.



Therefore Boy j partners with Girl $j+1$ if and only if Boy $j+1$ partners with Girl j. Therefore each permissible partnering can be visualised as some sequence of vertical lines and crosses, such as the one shown below. We have to count the number of such sequences.



If there are n crosses, where $n = 0, 1, 2, 3, 4, 5$, then there are $10-2n$ vertical lines, giving a total of $(10-2n)+n = 10-n$ symbols to arrange. Therefore there are n crosses to be arranged into $10-n$ positions. This can be done in $\binom{10-n}{n}$ ways.

So the number of ways the 10 boys and girls can be partnered is

$$\binom{10}{0}+\binom{9}{1}+\binom{8}{2}+\binom{7}{3}+\binom{6}{4}+\binom{5}{5} = 1+9+28+35+15+1 = \mathbf{89}.$$

7. *Method 1*

Since the decimal form of $1/n$ repeats every 8 digits, we can write

$$\frac{1}{n} = 0.abcdeghi\,abcdeghi\ldots$$

where a, b, c, d, e, g, h, i are decimal digits. Then

$$\frac{10^8}{n} = abcdeghi.abcdeghi\ldots$$

Subtracting gives

$$\begin{aligned}(10^8-1)\frac{1}{n} &= abcdeghi\\ 99\,999\,999 &= n \times abcdeghi\end{aligned}$$

Thus n is a factor of $99\,999\,999 = 9 \times 11 \times 101 \times 10001$.

We are told that the decimal expansion of $1/137$ has 8 digits repeating, so 137 must be a factor of $99\,999\,999$, hence a factor of 10001. So $99\,999\,999 = 3 \times 3 \times 11 \times 73 \times 101 \times 137$. Noting that 3, 11, 73, 101, 137 are all primes, the factors of $99\,999\,999$ in increasing order are 1, 3, 9, 11, 33, 73, ...

If $n = 3$, then $\frac{1}{n} = 0.3333\ldots$.
If $n = 9$, then $\frac{1}{n} = 0.1111\ldots$.
If $n = 11$, then $\frac{1}{n} = 0.0909\ldots$.
If $n = 33$, then $\frac{1}{n} = 0.0303\ldots$.
If $n = 73$, then $\frac{1}{n} = 0.01369863\,01369863\ldots$.
Thus the desired n is **73**.

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Method 2

If the decimal form of $1/n$ repeats every k digits, then we can write $\frac{1}{n} = 0.a_1a_2\ldots a_ka_1a_2\ldots a_k\ldots$, where $a_1, a_2, \ldots, a_k$ are decimal digits. Then

$$\begin{aligned} \frac{10^k}{n} &= a_1a_2\ldots a_k.a_1a_2\ldots a_k\ldots \\ (10^k-1)\frac{1}{n} &= a_1a_2\ldots a_k \\ 9\ldots 9 &= n \times a_1a_2\ldots a_k \end{aligned}$$

where $9\ldots9$ is the integer consisting of k 9s as its decimal digits. Thus n is a factor of $9\ldots9$.

Conversely, suppose n is a factor of $10^k - 1$. Then $(10^k-1)/n = a_1a_2\ldots a_k$ for some decimal digits $a_1, a_2, \ldots, a_k$. So we have

$$\begin{aligned} \frac{1}{n} &= 0.a_1a_2\ldots a_k + \frac{1}{n}\times 10^{-k} \\ &= 0.a_1a_2\ldots a_ka_1a_2\ldots a_k + \frac{1}{n}\times 10^{-2k} \\ &= 0.a_1a_2\ldots a_ka_1a_2\ldots a_ka_1a_2\ldots a_k + \frac{1}{n}\times 10^{-3k} \end{aligned}$$

and so on. Thus the decimal form of $1/n$ repeats every k digits.

If the decimal form of $1/n$ repeats every 8 digits but also repeats with a smaller number of digits, then that smaller number must be 1 or 2 or 4. So we want the smallest factor of 99 999 999 that is not a factor of any of 9, 99, 9999.

We have $99\,999\,999 = 9\times 11\times 101\times 10001$. We are told that the decimal expansion of $1/137$ has 8 digits repeating, so 137 must be a factor of 99 999 999, hence a factor of 10001. So $99\,999\,999 = 3\times 3\times 11\times 73\times 101\times 137$. Noting that 3, 11, 73, 101, 137 are all primes, the factors of 99 999 999 in increasing order are 1, 3, 9, 11, 33, 73, ...

Thus the desired n is **73**.

Method 3

As in Method 1, $n \times abcdeghi = 9999\,9999 = 9999 \times 10001$.

If 10001 divides $abcdeghi$, then n divides 9999. From Method 2, the decimal form of $1/n$ repeats every 4 digits, which is not permitted. So 10001 is not a factor of $abcdeghi$.

We are told that the decimal expansion of $1/137$ has 8 digits repeating, so 137 must be a factor of 9999 9999. Since 137 is prime and not a factor of 9999, it must be a factor of 10001. Since $10001 = 137\times 73$, at least one of 137 and 73 remains as a factor on the right side of $n\times abcdeghi = 9999\times 10001$ after both sides are divided by the greatest common divisor of the right side and $abcdeghi$. This implies that $n \geq 73$.

Since $\frac{1}{73} = 0.01369863\,01369863\ldots$, the desired n is **73**.

8. Since CAFDG, ABACDE, CHHBAED are side lengths of a triangle, we have
CAFDG + ABACDE > CHHBAED.

Since ABACDE > CHHBAED − CAFDG > 999 999 − 99 999 = 900 000, we have A = 9.

Since CHHBAED < ABACDE + CAFDG < 1000 000 + 100 000 = 1100 000,
we have C = 1, hence H = 0.

In summary we have 9B91DE > 100B9ED − 19FDG > 999 999 − 19 999 = 980 000.
So B ≥ 8. Since B ≠ A = 9, B = 8.

We have 19FDG > 10089ED − 9891DE > 1008 900 − 989 199 = 19 701.
So F ≥ 7. Since A = 9 and B = 8, F = 7.

We have 10089ED < 9891DE + 197DG.
To satisfy the 9 on the left side we need D ≥ 5.
If D = 5, then D + D =10 hence E = 0 or 1, which is impossible. So D = 6.

We have 10089E6 < 98916E + 1976G.
The only remaining digits are 2, 3, 4, 5. So E + G < 10.
Hence the second last digit in 98916E + 1976G is 2.
Therefore E in 10089E6 is at most 2.
So E = 2 and 1008926 < 989162 + 1976G.

Hence G = 5 and DEG = **625**.

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9. *Method 1*

We are given
$$x^3 - y^3 = 7(x - y) \qquad (1)$$
$$x^3 + y^3 = 5(x + y) \qquad (2)$$

First assume $x = y$. From (2), $2x^3 = 10x$, $x(x^2 - 5) = 0$, $x = 0$ or $x = \pm\sqrt{5}$.
So there are 3 solutions in this case: $(0, 0)$, $(\sqrt{5}, \sqrt{5})$, $(-\sqrt{5}, -\sqrt{5})$.
Next assume $x = -y$. From (1), $2x^3 = 14x$, $x(x^2 - 7) = 0$, $x = 0$ or $x = \pm\sqrt{7}$.
So there are 3 solutions in this case: $(0, 0)$, $(\sqrt{7}, -\sqrt{7})$, $(-\sqrt{7}, \sqrt{7})$.
Now assume that $x \neq \pm y$. Equations (1) and (2) become:
$$x^2 + xy + y^2 = 7 \qquad (3)$$
$$x^2 - xy + y^2 = 5 \qquad (4)$$

Adding (4) and (3) gives $x^2 + y^2 = 6$.
We show below that there are 4 real solutions in this case. So $n = \mathbf{9}$.
Therefore the sum of $x^2 + y^2$ over all nine real solutions (x, y) is
$$(0 + 0) + 2(5 + 5) + 2(7 + 7) + 4 \times 6 = 20 + 28 + 24 = \mathbf{72}.$$

Some alternative proofs that there are 4 solutions for equations (3) and (4).
1. Subtracting (4) from (3) gives $xy = 1$. Substitute $y = 1/x$ into equation (3).
$$\begin{aligned} x^2 + 1 + (1/x)^2 &= 7 \\ x^4 - 6x^2 + 1 &= 0 \\ x^2 &= (6 \pm \sqrt{32})/2 \\ &= 3 \pm 2\sqrt{2} \end{aligned}$$

Since $3 - 2\sqrt{2} > 0$, there are 4 real solutions.
2. We have $x^2 + y^2 = 6$ and $2xy = 2$. Therefore $(x + y)^2 = 8$ and $(x - y)^2 = 4$. Hence $x + y = \pm 2\sqrt{2}$ and $x - y = \pm 2$. The resulting 4 pairs of simultaneous equations give 4 solutions.
3. The graph of $x^2 + y^2 = 6$ on the Cartesian plane is a circle with radius $\sqrt{6}$ and its centre at the origin. The graph of $xy = 1$ is a hyperbola with its two branches in separate quadrants. The points (1,1) and $(-1, -1)$ are on the hyperbola and inside the circle since $\sqrt{2} < \sqrt{6}$. So each branch of the hyperbola intersects the circle at 2 points. This gives 4 solutions.

Method 2

We are given
$$x^3 - y^3 = 7(x - y) \qquad (1)$$
$$x^3 + y^3 = 5(x + y) \qquad (2)$$

By adding the two equations, we find that $y = 6x - x^3$.
Substituting in equation (2), ignoring the obvious solution $x = 0 = y$ and eventually substituting $t = x^2$, we have
$$\begin{aligned} x^3 + (6x - x^3)^3 &= 5x + 5(6x - x^3) = 35x - 5x^3 \\ x^2 + x^2(6 - x^2)^3 &= 35 - 5x^2 \\ t + t(6 - t)^3 &= 35 - 5t \\ t(216 - 108t + 18t^2 - t^3) &= 35 - 6t \\ t^4 - 18t^3 + 108t^2 - 222t + 35 &= 0 \end{aligned}$$

The sum of the roots for t is $-(-18) = 18$. If $t < 0$, then the left side of the last equation is positive. So $t > 0$, hence there are 2 values for x for each value of t. So assuming there are 4 distinct roots for t, the sum of x^2 over all real solutions (x, y) of the original equations is $2 \times 18 = 36$.

Since the original equations are symmetric in x and y, the sum of y^2 over all real solutions (x, y) of the original equations is also 36. So the sum of $x^2 + y^2$ over all real solutions (x, y) is $2 \times 36 = \mathbf{72}$.

To show that $Q(t) = t^4 - 18t^3 + 108t^2 - 222t + 35$ has 4 distinct roots, we use the remainder theorem. We have $Q(1) < 0$, $Q(3) < 0$, then $Q(5) = 0$ and $Q(7) = 0$. Dividing $Q(t)$ by $(t - 5)(t - 7) = t^2 - 12t + 35$ gives $t^2 - 6t + 1$. So $Q(t)$ has four distinct roots: 5, 7, $3 \pm 2\sqrt{2}$. So $n = \mathbf{9}$.

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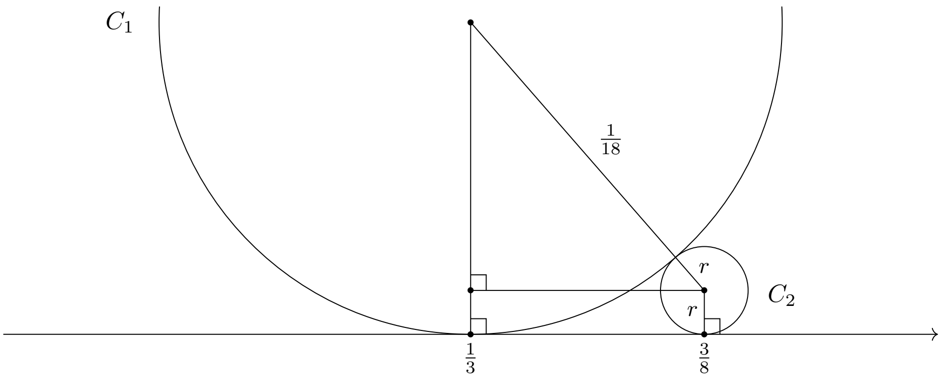
10. *Method 1*

Considering first the circles C_1 and C_2, let the radius of C_2 be r. Draw a right-angled triangle with sides $3/8-1/3 = 1/24$, $1/18-r$, and hypotenuse $1/18+r$ as shown.

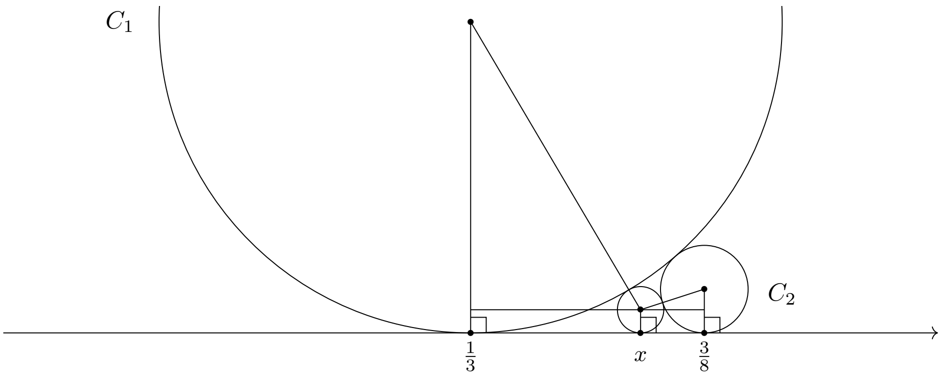


From Pythagoras we then obtain

$$\frac{1}{24^2} = \left(\frac{1}{18}+r\right)^2 - \left(\frac{1}{18}-r\right)^2 = \frac{2r}{9}$$

which simplifies to $r = 1/128$.

We now let s be the radius of C_3. Draw three vertical radii and two right-angled triangles as shown.



The right-angled triangle formed by C_1 and C_3 has sides $x-1/3$, $1/18-s$ and hypotenuse $1/18+s$. The right-angled triangle formed by C_2 and C_3 has sides $3/8-x$, $1/128-s$ and $1/128+s$. From Pythagoras in these two triangles we obtain

$$\left(x-\frac{1}{3}\right)^2 = \left(\frac{1}{18}+s\right)^2 - \left(\frac{1}{18}-s\right)^2 = \frac{2s}{9} \quad \text{so} \quad x-\frac{1}{3} = \frac{\sqrt{2s}}{3}, \quad \text{and}$$

$$\left(\frac{3}{8}-x\right)^2 = \left(\frac{1}{128}+s\right)^2 - \left(\frac{1}{128}-s\right)^2 = \frac{2s}{64} = \frac{s}{32} \quad \text{so} \quad \frac{3}{8}-x = \frac{\sqrt{2s}}{8}.$$

Hence $3x-1 = \sqrt{2s} = 3-8x$. Therefore $x = 4/11$ and $s = 1/(2\cdot 11^2) = 1/242$.

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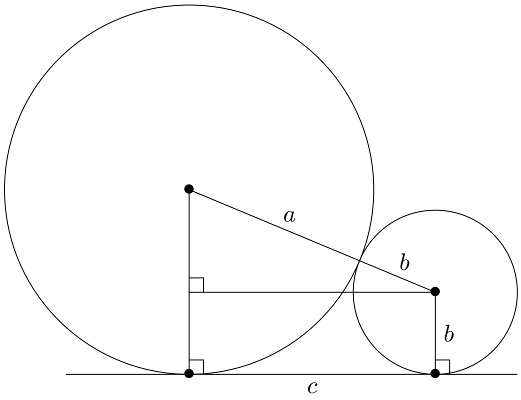
Method 2

Consider any two touching circles of radius a and b that are tangent to a line at points distance c apart.



Applying Pythagoras' theorem to the right-angled triangle in the diagram, we have
$c^2 = (a+b)^2 - (a-b)^2 = 4ab$, hence $c = 2\sqrt{ab}$.

With the notation in Method 1, this gives

$$x - \tfrac{1}{3} = 2\sqrt{s/18} = \sqrt{2s}/3 \qquad (1)$$
$$\tfrac{3}{8} - x = 2\sqrt{rs} \qquad (2)$$
$$\tfrac{3}{8} - \tfrac{1}{3} = 2\sqrt{r/18} = \sqrt{2r}/3 \qquad (3)$$

Multiplying (1) and (3), then using (2) gives

$$\begin{aligned}\tfrac{1}{24}(x - \tfrac{1}{3}) &= 2\sqrt{rs}/9\\ &= \tfrac{1}{9}(\tfrac{3}{8} - x)\\ 3(x - \tfrac{1}{3}) &= 8(\tfrac{3}{8} - x)\\ x &= 4/11\end{aligned}$$

From (1), $\sqrt{2s} = 3x - 1 = \frac{1}{11}$, hence $s = 1/242$.

Investigation

We are given $x_2 = 3/8$ and found above $x_3 = 4/11$. Also from above $r_2 = 1/128 = \frac{1}{2\cdot 8^2}$ and $r_3 = s = \frac{1}{2\cdot 11^2}$. We prove by induction that $x_n = \frac{n+1}{3n+2}$ and $r_n = \frac{1}{2(3n+2)^2}$ for all $n \geq 2$.

Consider the three circles C_1, C_n and C_{n+1}. We are given C_1 is tangent to the real line at $1/3$ and has radius $1/18$. Assuming $x_n = \frac{n+1}{3n+2}$ and $r_n = \frac{1}{2(3n+2)^2}$, we want to show that $x_{n+1} = \frac{n+2}{3n+5}$ and $r_{n+1} = \frac{1}{2(3n+5)^2}$.

Setting up two right-angled triangles as above, but now with circles C_1, C_n, C_{n+1}, two applications of Pythagoras give

$$\left(x_{n+1} - \frac{1}{3}\right)^2 = \left(\frac{1}{18} + r_{n+1}\right)^2 - \left(\frac{1}{18} - r_{n+1}\right)^2 = \frac{2r_{n+1}}{9}$$
$$\left(\frac{n+1}{3n+2} - x_{n+1}\right)^2 = \left(\frac{1}{2(3n+2)^2} + r_{n+1}\right)^2 - \left(\frac{1}{2(3n+2)^2} - r_{n+1}\right)^2 = \frac{2r_{n+1}}{(3n+2)^2}$$

From these two equations we obtain

$$3x_{n+1} - 1 = \sqrt{2r_{n+1}} = n + 1 - (3n+2)x_{n+1}$$

which yields $x_{n+1} = \frac{n+2}{3n+5}$. Then the first equality yields $r_{n+1} = \frac{1}{2(3n+5)^2}$.

So we have proved by induction that, for all $n \geq 2$, $x_n = \frac{n+1}{3n+2}$ and $r_n = \frac{1}{2(3n+2)^2}$.

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Comments

1. It turns out that $x_n = \frac{n+1}{3n+2}$ is in simplest form for all $n \geq 2$. To see why, suppose that $n+1$ and $3n+2$ have a common factor d, then d divides $3(n+1) - (3n+2) = 1$. Hence $d = 1$.

2. The numerators and denominators of consecutive fractions x_n and x_{n+1} satisfy a curious equation. If $a_n = n+1$ and $b_n = 3n+2$, then

$$a_n b_{n+1} - a_{n+1} b_n = (n+1)(3n+5) - (n+2)(3n+2) = 1.$$

For example, $x_2 = 3/8$, $x_3 = 4/11$, and $3 \times 11 - 4 \times 8 = 33 - 32 = 1$.

3. The numerators and denominators of consecutive fractions x_n, x_{n+1}, x_{n+2}, satisfy another curious equation. If $a_n = n+1$ and $b_n = 3n+2$, then

$$\frac{a_n + a_{n+2}}{b_n + b_{n+2}} = \frac{(n+1)+(n+3)}{(3n+2)+(3n+8)} = \frac{2n+4}{6n+10} = \frac{n+2}{3n+5} = \frac{a_{n+1}}{b_{n+1}}$$

4. The circles in this problem are examples of *Ford circles*. A Ford circle is a circle $C[a,b]$ whose centre is $(\frac{a}{b}, \frac{1}{2b^2})$, where the greatest common divisor of a and b is 1.

It turns out that two Ford circles $C[a,b]$ and $C[c,d]$ are tangent if $ad - bc = \pm 1$ and disjoint otherwise. For example, as we saw above, $C[1,3]$ is tangent to $C[3,8]$ and $1 \times 8 - 3 \times 3 = -1$, $C[3,8]$ is tangent to $C[4,11]$ and $3 \times 11 - 4 \times 8 = 1$, but $C_2 = C[3,8]$ is disjoint from $C_4 = C[5,14]$ and $3 \times 14 - 5 \times 8 = 2 \neq \pm 1$.

It is also true that $C[a,b]$ and $C[c,d]$ are both tangent to $C[a+c, b+d]$.
For example, $C[1,3]$ and $C[3,8]$ are tangent to $C[4,11]$, and $C[a_n, b_n]$ and $C[a_{n+2}, b_{n+2}]$ are tangent to $C[a_n + a_{n+2}, b_n + b_{n+2}] = C[a_{n+1}, b_{n+1}]$.

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Statistics

Distribution of awards/school year

School Year	Number of Students	Prize	High Distinction	Distinction	Credit	Participation
8	475	8	27	65	109	266
9	530	4	48	73	158	247
10	577	15	61	90	159	252
Other	321	2	17	36	61	205
All Years	**1903**	**29**	**153**	**264**	**487**	**970**

Number of correct answers for questions 1 to 8

School Year	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
8	285	243	325	93	141	**77**	**86**	128
9	324	353	395	166	131	**102**	**98**	156
10	332	382	409	221	191	**117**	**123**	186
Other	161	151	197	62	83	**41**	**49**	75
All Years	**1102**	**1129**	**1326**	**542**	**546**	**337**	**356**	**545**

Mean score/question/school year

School Year	Number of Students	Q1-8	Q9	Q10	Overall Mean
8	475	8.5	0.3	0.2	9.1
9	530	9.4	0.5	0.5	10.4
10	577	10.1	0.7	0.6	11.4
Other	321	7.4	0.2	0.2	7.8
All Years	**1903**	**9.0**	**0.5**	**0.4**	**9.9**