

Problems

Time allowed: 4 hours.

NO calculators are to be used.

Questions 1 to 8 only require their numerical answers all of which are non-negative integers less than 1000.

Questions 9 and 10 require written solutions which may include proofs.

The bonus marks for the Investigation in Question 10 may be used to determine prize winners.

1. Two positive integers, a and b , satisfy the equation

$$\frac{2^{24} + 2^{21} + 2^{21} + 2^{21}}{2024} = \frac{2^a}{23^b}.$$

Find the value of $a + b$.

[2 marks]

2. The area of a right-angled triangle is 120 and each of its three sides has integer length.

Find the length of its hypotenuse.

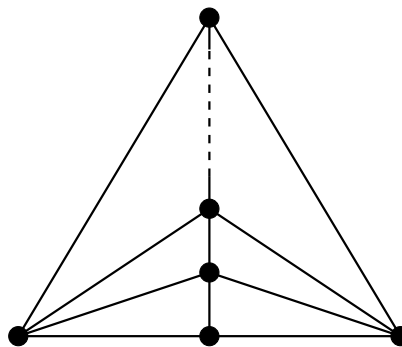
[2 marks]

3. Quadrilateral $ABCD$ has perimeter 60 cm and AB has length 9 cm. Sides AD and BC are parallel with distance 7 cm between them. Points E and F are chosen on sides BC and AD respectively so that the line EF divides $ABCD$ into two quadrilaterals with equal perimeters and equal areas.

What is the area of quadrilateral $ABCD$?

[3 marks]

4. In the following figure, all the central dots are collinear and the bottom three dots are collinear. There are exactly 2024 triangles that have dots as vertices (no three collinear dots form a triangle).



How many dots are there?

[3 marks]

5. Three positive integers a, b, c satisfy

$$a^2 + b - c = 100 \tag{1}$$

$$a + b^2 - c = 124 \tag{2}$$

Find $a + b + c$.

[3 marks]

6. Let $ABCD$ be a parallelogram in which the bisector of angle ABC intersects AD at P .
Given $PD = 65$ and $BP = 78 = PC$, find AB .

[4 marks]

7. A survey of N people was taken to determine which of the different types A, B, C of screen entertainment they used, if any. The survey found:
- 50 people used B
 - 61 did not use A
 - 13 did not use C
 - 74 used at least two of A, B, C.

Find the minimum value of N .

[4 marks]

8. A *binary sequence* is a sequence in which each term is 0 or 1.
A binary sequence $(a_0, a_1, \dots, a_{n-1}, a_n)$ is *dyadic* if it has an even number of terms and
 $a_0 + a_n = a_1 + a_{n-1} = a_2 + a_{n-2} = \dots = 1$.
For example, $(1, 0, 0, 1, 1, 0)$ is a dyadic sequence of length 6.

How many binary sequences are there of length 16 that are *not* dyadic, but are composed of 4 consecutive dyadic sequences each of length 4?

[4 marks]

9. Denote the number of positive factors of an integer n by $F(n)$.
For example, the positive factors of 12 are 1, 2, 3, 4, 6, 12, so $F(12) = 6$.
In addition, the positive factors of 6 are 1, 2, 3, 6, so $F(F(12)) = 4$.
If $F(F(n)) = 3$ and $F(F(6n)) = 7$, find $F(n)$.

[5 marks]

10. Suppose that a number of 5×7 rectangles are placed on a large grid of unit squares so that each rectangle covers exactly 35 of the unit squares. The rectangles can be placed either horizontally or vertically and they may overlap.

Find the largest integer N for which it is *not* possible to cover exactly N unit squares in this way.

[5 marks]

Investigation

Suppose instead we have rectangles of a single size $3 \times n$, for some positive integer $n \geq 6$. Placing them on the grid as above, what is the largest integer N for which it is *not* possible to cover exactly N unit squares?

[4 bonus marks]

Solutions

1. We have

$$\frac{2^a}{23^b} = \frac{2^{24} + 3 \times 2^{21}}{2024} = \frac{2^{21}(2^3 + 3)}{2^3 \times 11 \times 23} = \frac{2^{21} \times 11}{2^3 \times 11 \times 23} = \frac{2^{18}}{23}.$$

Hence $23 \times 2^a = 23^b \times 2^{18}$. Since 2 and 23 are primes and a and b are positive integers, $b = 1$ and $a = 18$. Therefore $a + b = \mathbf{19}$.

2. *Method 1*

Let x, y, z be the side lengths of the triangle with $x \leq y < z$. From Pythagoras we have $x^2 + y^2 = z^2$, and we are given $(xy)/2 = 120$. Since $xy = 240 = 2^4 \times 3 \times 5$, the ordered pair (x, y) is one or more of (1,240), (2,120), (3,80), (4,60), (5,48), (6,40), (8,30), (10,24), (12,20), (15,16).

For the first six pairs, we calculate $(y + 1)^2 - y^2 = 2y + 1$. If that exceeds x^2 , then $x^2 + y^2$ is not a square. For the remaining pairs, we calculate $x^2 + y^2$ and check directly if it is square.

x	y	$2y + 1 > x^2?$	$x^2 + y^2$ square?
1	240	yes	No
2	120	yes	No
3	80	yes	No
4	60	yes	No
5	48	yes	No
6	40	yes	No
		$x^2 + y^2$	
8	30	964	No
10	24	676	Yes ($z = 26$)
12	20	544	No
15	16	481	No

Thus the length of the hypotenuse is **26**.

Comment

We could divide the (x, y) pairs by their highest common factors, giving (1,240), (1,60), (3,80), (1,15), (5,48), (3,20), (4,15), (5,12), (3,5), (15,16), and then recall that (5,12) is the only pair in this list that gives a Pythagorean triple.

Alternatively, we could list familiar pythagorean triples until we find one that gives a triangle of area 120: (3,4,5), (6,8,10), (9,12,15), (12,16,20), (5,12,13), (10,24,26). The last suffices but is it unique?

Method 2

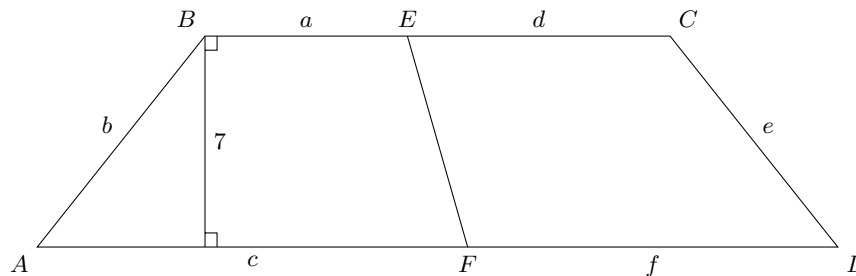
Let x, y, z be the side lengths of the triangle with $x \leq y < z$. From Pythagoras we have $x^2 + y^2 = z^2$, and we are given $xy = 240$. So $(x + y)^2 = z^2 + 480$ and $(x - y)^2 = z^2 - 480$. Let $r = x + y$. Then $480 = r^2 - z^2 = (r + z)(r - z)$. Since $(r + z) - (r - z) = 2z$, both factors are even. Since $z^2 \geq 480$, the ordered pair $(r - z, r + z)$ is one or more of $(2, 140)$, $(4, 120)$, $(6, 80)$, $(8, 60)$.

For each of the ordered pairs, we calculate z and check if $z^2 - 480$ is a square.

$r - z$	$r + z$	z	$z^2 - 480$	$z^2 - 480$ square?
2	140	69	4281	No (between 65^2 and 66^2)
4	120	58	2884	No (between 53^2 and 54^2)
6	80	37	889	No (between 29^2 and 30^2)
8	60	26	196	yes ($=14^2$)

Thus the length of the hypotenuse is **26**.

3. Quadrilateral $ABCD$ is shown below with the perimeter divided into segments of lengths a, b, c, d, e, f .



From the information on perimeter, we know that $a + b + c = d + e + f$.

From the information on area, we know that $7(c + a)/2 = 7(d + f)/2$, hence $c + a = d + f$.

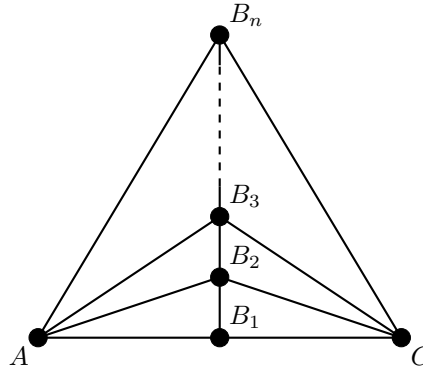
Therefore $e = b = 9$.

Since $ABCD$ is a trapezium, its area is $7(a + d + c + f)/2$.

We are given $a + b + c + d + e + f = 60$ and we know that $b + e = 18$.

Hence $a + d + c + f = 42$ and the area of the $ABCD$ is $(7 \times 42)/2 = 147 \text{ cm}^2$.

4. Label the dots as shown.



Method 1

There are three types of triangle:

1. AB_iC where $2 \leq i \leq n$: there are $n - 1$ triangles of this type.
2. AB_iB_j where $1 \leq i < j \leq n$: the number of triangles of this type equals the number of ways to choose the ordered pair (i, j) , namely $\binom{n}{2} = \frac{1}{2}n(n - 1)$.
(Alternatively, there are $n - i$ triangles of this type for each value of i .
So the number of these triangles is $(n - 1) + (n - 2) + \dots + 2 + 1 = \frac{1}{2}n(n - 1)$.)
3. CB_iB_j , where $1 \leq i < j \leq n$: the number of triangles of this type is the same as the number of triangles of type 2, namely $\frac{1}{2}n(n - 1)$.

Therefore the total number of triangles is

$$(n - 1) + \frac{1}{2}n(n - 1) + \frac{1}{2}n(n - 1) = (n - 1) + n(n - 1) = (n - 1)(n + 1) = n^2 - 1.$$

Solving $n^2 - 1 = 2024$, we have $n^2 = 2025$, hence $n = 45$.

So the number of dots is $n + 2 = \mathbf{47}$.

Method 2

All triples of dots form a triangle except for the bottom three dots and the triples of dots on the central line.

So we have

$$\begin{aligned} 2024 &= \binom{n+2}{3} - \binom{n}{3} - 1 \\ 6 \times 2024 &= (n+2)(n+1)n - n(n-1)(n-2) - 6 \\ 6 \times 2025 &= n(n^2 + 3n + 2) - n(n^2 - 3n + 2) \\ &= 6n^2 \\ n &= 45 \end{aligned}$$

Hence the number of dots is $n + 2 = \mathbf{47}$.

5. Subtract (1) from (2)

$$\begin{aligned} 124 - 100 &= (a + b^2 - c) - (a^2 + b - c) \\ 24 &= a + b^2 - c - a^2 - b + c \\ &= (b^2 - a^2) + (a - b) \\ &= (b - a)(b + a) - (b - a) \\ &= (b - a)(b + a - 1) \end{aligned}$$

Since $b + a - 1$ is positive, $b - a$ must also be positive.

Moreover, $b - a$ is the smaller factor since $(b + a - 1) - (b - a) = 2a - 1 > 0$.

So $b - a = 1, 2, 3$, or 4 .

If $b - a = 4$, then $b + a - 1 = 6$. Adding gives $2b = 11$, which is impossible.

If $b - a = 3$, then $b + a - 1 = 8$. Adding gives $2b = 12$, hence $b = 6$, $a = 3$, and $c = 9 + 6 - 100$, which is negative.

If $b - a = 2$, then $b + a - 1 = 12$. Adding gives $2b = 15$, which is impossible.

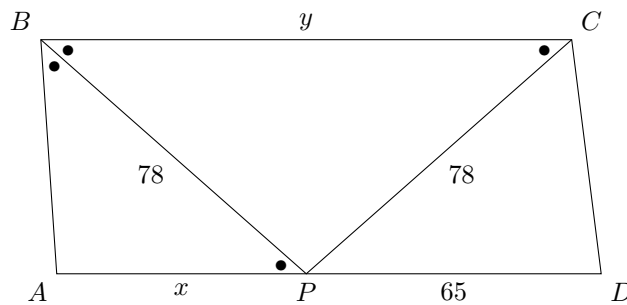
If $b - a = 1$, then $b + a - 1 = 24$. Adding gives $2b = 26$, hence $b = 13$, $a = 12$, and $c = 144 + 13 - 100 = 57$.

Therefore $a + b + c = 12 + 13 + 57 = \mathbf{82}$.

Comment

Since $(b - a) + (b + a - 1) = 2b - 1$, which is odd, $b - a$ and $b + a - 1$ have opposite parity. So the pairs $(2, 12)$ and $(4, 6)$ can be eliminated immediately.

6. Since BP bisects $\angle ABC$ and $BC \parallel AD$ and $\triangle BPC$ is isosceles, the dotted angles are equal.



Since $BC = AD$, $y = x + 65$.

Method 1

Since triangles BPA and BPC are equiangular, they are similar.

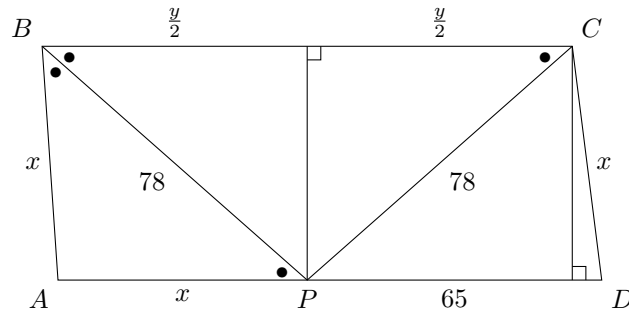
Hence

$$\begin{aligned} \frac{78}{y} &= \frac{x}{78} \\ 78^2 &= xy = x(x + 65) \\ 0 &= x^2 + 5 \times 13x - 6^2 \times 13^2 \\ &= (x + 9 \times 13)(x - 4 \times 13) \end{aligned}$$

Since $x > 0$, $x = 4 \times 13 = 52$. Since $\triangle BAP$ is isosceles, $AB = x = \mathbf{52}$.

Method 2

Since $\triangle BAP$ is isosceles, $AB = x$. Since $ABCD$ is a parallelogram, $CD = x$.
Draw lines through P and C perpendicular to BC and AD as indicated.

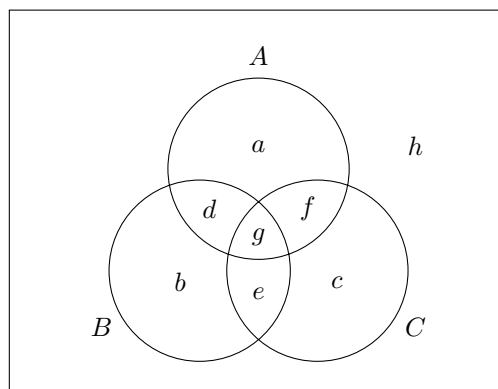


Applying Pythagoras' theorem to the appropriate right-angled triangles gives

$$\begin{aligned} 78^2 - \left(\frac{y}{2}\right)^2 &= x^2 - \left(65 - \frac{y}{2}\right)^2 \\ &= x^2 - 65^2 - \left(\frac{y}{2}\right)^2 + 65y \\ 78^2 &= x^2 - 65^2 + 65\left(x + 65\right) \\ 0 &= x^2 + 5 \times 13x - 6^2 \times 13^2 \\ &= (x + 9 \times 13)(x - 4 \times 13) \end{aligned}$$

Since $x > 0$, $x = 4 \times 13 = 52$. So $AB = x = \mathbf{52}$.

7. Consider the following Venn diagram representing the number of people using various combinations of entertainment types A, B, C.



We have

$$b + d + e + g = 50 \tag{1}$$

$$b + e + c + h = 61 \tag{2}$$

$$a + b + d + h = 13 \tag{3}$$

$$d + e + f + g = 74 \tag{4}$$

Method 1

Now

$$\begin{aligned} N &= (d + e + f + g) + a + (b + c + h) \\ &= 74 + a + (61 - e) \\ &= 74 + a + 61 - (50 - b - d - g) \\ &= 85 + a + b + d + g \geq 85 \end{aligned}$$

If $N = 85$, then $a = b = d = g = 0$. From (1), $e = 50$. From (3), $h = 13$. This implies $b + e + c + h \geq 63$, which contradicts (2). So $N \geq 86$.

If $N = 86$, then we have 5 equations in 8 variables. To see if these have a solution try letting $a = b = c = 0$ and then solving the 5 equations in the remaining 5 variables. Elimination gives $d = 1, e = 49, f = 24, g = 0, h = 12$. Since $a + b + c + d + e + f + g + h = 86$, we have found a solution which shows that the minimum value of N is **86**.

Method 2

We find each of d, e, f, h in terms of a, b, c, g .

Calculating (1) + (3) - (2) gives $2d = 2 - a - b + c - g$.

Calculating (1) + (2) - (3) gives $2e = 98 + a - b - c - g$.

Calculating (4) - (1) gives $f = 24 + b$.

Calculating (2) + (3) - (1) gives $2h = 24 - a - b - c + g$.

From this we have

$$\begin{aligned} N &= a + b + c + g + (d + e + h) + f \\ &= a + b + c + g + (124 - a - 3b - c - g)/2 + 24 + b \\ &= 86 + (a + b + c + g)/2 \geq 86 \end{aligned}$$

If $a = b = c = g = 0$, then $d = 1, e = 49, f = 24, h = 12$, and $N = 86$. Hence the minimum value of N is **86**.

Comment

Similar calculations show that the maximum value of N is 142.

8. A dyadic sequence of length 4 must have the form $(a, b, 1 - b, 1 - a)$ where each of a and b is 0 or 1. There are 2 choices for a and 2 independent choices for b . So the number of dyadic sequences of length 4 is $2 \times 2 = 4$.

Hence the number of sequences of length 16 composed of 4 consecutive dyadic sequences of length 4, is $4 \times 4 \times 4 \times 4 = 256$.

Of these sequences, those that are dyadic must have the form

$$(a, b, 1 - b, 1 - a, c, d, 1 - d, 1 - c, c, d, 1 - d, 1 - c, a, b, 1 - b, 1 - a)$$

where each of a, b, c, d is 0 or 1. So the number of sequences of this type is $2 \times 2 \times 2 \times 2 = 16$.

So the number of non-dyadic sequences of length 16 that are composed of 4 consecutive dyadic sequences of length 4, is $256 - 16 = \mathbf{240}$.

9. Recall that if $n = p_1^{m_1} p_2^{m_2} \dots$, where p_1, p_2, \dots are distinct primes, then $F(n) = (m_1 + 1)(m_2 + 1) \dots$. From this we also know that $F(n)$ is odd if and only if n is a square.

The only integers with 3 factors are squares of primes. Therefore, as $F(F(n)) = 3$, we have $F(n) = p^2$ for some prime number p . In addition, the only integers with 7 factors are sixth powers of primes. Since $F(F(6n)) = 7$, we have $F(6n) = q^6$ for some prime number q .

Method 1

Since $F(n) = p^2$, we have $n = t^{p^2-1}$ where t is prime, or $n = r^{p-1} \times s^{p-1}$ where r and s are primes.

Case 1. If $n = t^{p^2-1}$, then $6n = 2 \times 3 \times t^{p^2-1}$.

If $t = 2$ or 3 , then $6n$ would have $2 \times (p^2 + 1)$ factors. This means $2 \times (p^2 + 1) = q^6$, hence $q = 2$. So $p^2 + 1 = 2^5$ and $p^2 = 31$, which is impossible.

If $t \neq 2$ and $t \neq 3$, then $6n$ would have $2 \times 2 \times p^2$ factors. This means $2 \times 2 \times p^2 = q^6$, hence $q = 2$. So $p^2 = 2^4$ and $p = 4$, which is also impossible.

Case 2. If $n = r^{p-1} \times s^{p-1}$, then $6n = 2 \times 3 \times r^{p-1} \times s^{p-1}$.

If $r \neq 2$ and $s \neq 3$, then $6n$ would have $2 \times 2 \times p \times p = (2p)^2$ factors. This means $2^2 p^2 = q^6$, hence $2p = q^3$, which is impossible.

If $r = 2$ and $s \neq 3$, then $6n = 3 \times 2^p \times s^{p-1}$. This means $2 \times (p + 1) \times p = q^6$, hence $q = 2$. So $(p + 1) \times p = 32$, which is impossible.

If $r \neq 2$ and $s = 3$, then we obtain the same contradiction.

Therefore $r = 2$ and $s = 3$ and $6n = 2^p \times 3^p$. This means $(p + 1)^2 = q^6$ and $p + 1 = q^3$. If q is odd, then p is even. This means $p = 2$, hence $q^3 = 3$, which is impossible. Therefore $q = 2$, $p = 7$, $n = 2^6 \times 3^6$, and $F(n) = 7 \times 7 = \mathbf{49}$.

Method 2

Since n and $6n$ cannot both be squares, at least one of $F(n)$ and $F(F(6n))$ must be even. Therefore $p = 2$ or $q = 2$.

If $p = 2$, then $F(n) = 4$. Hence $n = r^3$, where r is prime, or $n = st$, where s and t are distinct primes with $s < t$. So $6n$ is one of $6r^3$ ($r \neq 2, 3$), 2×3^4 , 3×2^4 , $6st$ ($s \neq 2, t \neq 3$), $2 \times 3^2 s$ ($s \neq 2$), $3 \times 2^2 t$ ($t \neq 3$), $2^2 \times 3^2$. Hence $F(6n) \leq 16$. This contradicts $q^6 \geq 2^6 = 64$. Therefore p is odd. So $q = 2$ and $F(6n) = 64$.

If n has a prime factor that is neither 2 nor 3, then the greatest common divisor of $F(n)$ and $F(6n)$ would be greater than 1. This is impossible since the only prime factor of $F(n)$ is p , which is odd, and the only prime factor of $F(6n)$ is 2. Therefore $n = 2^a 3^b$.

So $F(n) = (a + 1)(b + 1) = p^2$ and $F(6n) = (a + 2)(b + 2) = 64$. From the second equation, the pair (a, b) is one or more of $(0, 30), (2, 14), (6, 6), (14, 2), (30, 0)$. Of these, only $(6, 6)$ satisfies the first equation. Hence $F(n) = \mathbf{49}$.

10. We place 5×7 rectangles with their length horizontal. Starting with one rectangle in each case, we produce five infinite families of coverings:
1. By overlaying the original rectangle with one shifted right 2 units, we cover precisely 45 unit squares; then adding rectangles shifted to the right, we can cover $45 + 5k$ unit squares for any $k \geq 0$. These are all the numbers from 45 that have a remainder of 0 when divided by 5.
 2. By overlaying the original rectangle with one shifted both right 1 unit and up 1 unit, we cover precisely 46 unit squares; then adding rectangles shifted to the right, we can cover $46 + 5k$ unit squares for any $k \geq 0$. These are all the numbers from 46 that have a remainder of 1 when divided by 5.
 3. By overlaying the original rectangle with one shifted right 1 unit and another shifted up 1 unit, we cover precisely 47 unit squares; then adding rectangles shifted to the right, we can cover $47 + 5k$ unit squares for any $k \geq 0$. These are all the numbers from 47 that have a remainder of 2 when divided by 5.
 4. By overlaying the original rectangle with one shifted up 1 unit, one shifted right 1 unit, and one shifted both right 1 unit and up 1 unit, we cover precisely 48 squares; then adding rectangles shifted to the right, we can cover $48 + 5k$ unit squares for any $k \geq 0$. These are all the numbers from 48 that have a remainder of 3 when divided by 5.
 5. By overlaying the original rectangle with one shifted up 2 units, we cover precisely 49 unit squares; then adding rectangles shifted to the right, we can cover $49 + 5k$ unit squares for any $k \geq 0$. These are all the numbers from 49 that have a remainder of 4 when divided by 5.

Each number greater than 44 is in one of these families according to its remainder when divided by 5. Hence there is a covering of any number of squares greater than 44. We now show that it is impossible to cover exactly 44 squares.

Let us assume that exactly 44 squares can be covered. This requires at least two rectangles. If they are placed in different orientations, then the number of squares covered by both is at most $5 \times 5 = 25$. Hence, the total number of covered squares is at least $35 + 35 - 25 = 45$, a contradiction. If two rectangles are placed in the same orientation, with one shifted both horizontally and vertically with respect to the other, then the number of squares covered by both is at most $4 \times 6 = 24$. Hence, the total number of covered squares is at least $35 + 35 - 24 = 46$, another contradiction.

So, to cover exactly 44 squares, the rectangles must all be horizontal translations of each other or all are vertical translations of each other. But the number of squares covered in this way must be a multiple of 5 or 7, a contradiction. Therefore, the largest value of N for which it is not possible to cover exactly N squares is **44**.

Investigation

Method 1

Start with a $3 \times n$ rectangle. We produce three infinite families of coverings:

1. By overlaying the original rectangle with one shifted both across 1 unit and down one unit, we cover precisely $4n + 2$ unit squares; then adding rectangles shifted to the right, we can cover $4n + 2 + 3k$ unit squares for any $k \geq 0$. These are all the numbers from $4n + 2$ that are congruent to $n + 2$ modulo 3.
2. By overlaying the original rectangle with one shifted down 1 unit and one shifted across 1 unit, we cover precisely $4n + 3$ unit squares; then adding rectangles shifted to the right, we can cover $4n + 3 + 3k$ unit squares for any $k \geq 0$. These are all the numbers from $4n + 3$ that are congruent to n modulo 3.
3. By overlaying the original rectangle one shifted down 1 unit, one shifted across 1 unit, and one shifted both across 1 and down 1 unit, we cover precisely $4(n + 1) = 4n + 4$ squares; then adding rectangles shifted to the right, we can cover $4n + 4 + 3k$ unit squares for any $k \geq 0$. These are all the numbers from $4n + 4$ that are congruent to $n + 1$ modulo 3.

Since $n, n + 1, n + 2$ are congruent modulo 3 to 0, 1, 2, in some order for any n , each number greater than $4n + 1$ is in one of the families according to its remainder when divided by 3. Hence there is a covering of any number of squares greater than $4n + 1$. In addition, $4n$ squares are covered by two overlapping rectangles with one shifted down one unit. Also, if n is congruent to 2 modulo 3, then $4n + 1$ is divisible by 3 so $4n + 1$ squares can be covered with some rectangles overlapping horizontally.

We now claim that when $n \equiv 0$ or $1 \pmod{3}$, the number of unit squares covered cannot be $4n + 1$. Suppose to the contrary that some arrangement of rectangles covers precisely $4n + 1$ unit squares. If two rectangles are placed in different orientations, then the number of squares covered by both is at most $3 \times 3 = 9$. Hence, the total number of covered squares is at least $3n + 3n - 9 = 6n - 9$. But as $n > 5$ it follows that $6n - 9 > 4n + 1$, a contradiction. Thus all $3 \times n$ rectangles have the same orientation. If two rectangles are placed in the same orientation, with one shifted both horizontally and vertically with respect to the other, then the number of squares covered by both is at most $2(n - 1)$. Hence, the total number of covered squares is at least $3n + 3n - 2n + 2 = 4n + 2 > 4n + 1$, a contradiction. Hence, if rectangles cover exactly $4n + 1$ squares, then they must all be horizontal translations of each other or all are vertical translations of each other. But the number of squares covered in this way must be a multiple of 3 or n . This is impossible since $4n + 1 \equiv n + 1 \equiv 1$ or $2 \pmod{3}$ and $4n + 1 \equiv 1 \pmod{n}$.

We next claim that when $n \equiv 2 \pmod{3}$, the number of unit squares covered cannot be $4n - 1$. Suppose to the contrary that some number of rectangles covers precisely $4n - 1$ unit squares. As above, the rectangles must all be horizontal translations of each other or all are vertical translations of each other, hence the number of squares covered must be a multiple of 3 or n . This is impossible since $4n - 1 \equiv n - 1 \equiv 1 \pmod{3}$ and $4n - 1 \equiv n - 1 \pmod{n}$.

We conclude that when $n \equiv 0$ or $1 \pmod{3}$ that the largest integer N for which it is not possible to cover exactly N unit squares is $4n + 1$, and when $n \equiv 2 \pmod{3}$, this N is $4n - 1$.

Method 2

Start with a $3 \times n$ rectangle. We produce four infinite families of coverings:

1. By adding rectangles shifted to the right, we can cover $3n + 3k$ unit squares for any $k \geq 0$.
2. By overlaying the original rectangle with one shifted both across 1 unit and down one unit, we cover precisely $4n + 2$ unit squares; then adding rectangles shifted to the right, we can cover $4n + 2 + 3k$ unit squares for any $k \geq 0$.
3. By overlaying the original rectangle with one shifted down 1 unit, we cover precisely $4n$ unit squares; then adding rectangles shifted to the right, we can cover $4n + 3k$ unit squares for any $k \geq 0$.
4. By overlaying the original rectangle one shifted down 1 unit, one shifted across 1 unit, and one shifted both across 1 and down 1 unit, we cover precisely $4(n + 1) = 4n + 4$ squares; then adding rectangles shifted to the right, we can cover $4n + 4 + 3k$ unit squares for any $k \geq 0$.

Note that $3n + 3k \equiv 0 \pmod{3}$, $4n + 2 + 3k \equiv n + 2 \pmod{3}$, $4n + 3k \equiv n \pmod{3}$, and $4n + 4 + 3k \equiv n + 1 \pmod{3}$.

Now we consider the congruence classes of $n \pmod{3}$ and consider what numbers of unit squares can be covered in each case.

1. If $n \equiv 0 \pmod{3}$ then $3n + 3k$ covers all numbers $\equiv 0 \pmod{3}$ from $3n$ upwards (including $4n + 3$ upwards); $4n + 4 + 3k$ covers all numbers $\equiv 1 \pmod{3}$ from $4n + 4$ upwards; and $4n + 2 + 3k$ covers all numbers $\equiv 2 \pmod{3}$ from $4n + 2$ upwards. Thus the above constructions cover all numbers of squares from $4n + 2$ upwards.
2. If $n \equiv 1 \pmod{3}$ then $3n + 3k$ covers all numbers $\equiv 0 \pmod{3}$ from $3n$ upwards (including $4n + 2$ upwards); $4n + 3k$ covers all numbers $\equiv 1 \pmod{3}$ from $4n$ upwards (including $4n + 3$ upwards); and $4n + 4 + 3k$ covers all numbers $\equiv 2 \pmod{3}$ from $4n + 4$ upwards. Thus the above constructions cover all numbers of squares from $4n + 2$ upwards.
3. If $n \equiv 2 \pmod{3}$ then $3n + 3k$ covers all numbers $\equiv 0 \pmod{3}$ from $3n$ upwards (including $4n + 1$ upwards); $4n + 2 + 3k$ covers all numbers $\equiv 1 \pmod{3}$ from $4n + 2$ upwards; and $4n + 3k$ covers all numbers $\equiv 2 \pmod{3}$ from $4n$ upwards. Thus the above constructions cover all numbers of squares from $4n$ upwards.

We now claim that when $n \equiv 0$ or $1 \pmod{3}$, the number of unit squares covered cannot be $4n + 1$. Suppose to the contrary that some arrangement of rectangles covers precisely $4n + 1$ unit squares. If two rectangles are placed in different orientations, then the number of squares covered by both is at most $3 \times 3 = 9$. Hence, the total number of covered squares is at least $3n + 3n - 9 = 6n - 9$. But as $n > 5$ it follows that $6n - 9 > 4n + 1$, a contradiction. Thus all $3 \times n$ rectangles have the same orientation. If two rectangles are placed in the same orientation, with one shifted both horizontally and vertically with respect to the other, then the number of squares covered by both is at most $2(n - 1)$. Hence, the total number of covered squares is at least $3n + 3n - 2n + 2 = 4n + 2 > 4n + 1$, a contradiction. Hence, if rectangles cover exactly $4n + 1$ squares, then they must all be horizontal translations of each other or all are vertical translations of each other. But the number of squares covered in this way must be a multiple of 3 or n . This is impossible since $4n + 1 \equiv n + 1 \equiv 1$ or $2 \pmod{3}$ and $4n + 1 \equiv 1 \pmod{n}$.

We next claim that when $n \equiv 2 \pmod{3}$, the number of unit squares covered cannot be $4n - 1$. Suppose to the contrary that some number of rectangles covers precisely $4n - 1$ unit squares. As above, the rectangles must all be horizontal translations of each other or all are vertical translations of each other, hence the number of squares covered must be a multiple of 3 or n . This is impossible since $4n - 1 \equiv n - 1 \equiv 1 \pmod{3}$ and $4n - 1 \equiv n - 1 \pmod{n}$.

We conclude that when $n \equiv 0$ or $1 \pmod{3}$ that the largest integer N for which it is not possible to cover exactly N unit squares is $4n + 1$, and when $n \equiv 2 \pmod{3}$, this N is $4n - 1$.

2024 Australian Intermediate Mathematics Olympiad Statistics

Distribution of Awards/School Year

School Year	Number of Students	Prize	High Distinction	Distinction	Credit	Participation
8	573	5	25	61	166	316
9	724	8	65	92	212	347
10	725	18	70	114	231	292
Other	456	1	23	39	94	299
All Years	2478	32	183	306	703	1254

Number of Correct Answers for Questions 1 to 8

School Year	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
8	280	410	4	6	370	68	0	0
9	400	539	5	15	479	129	0	0
10	467	548	10	26	450	151	0	0
Other	168	273	2	7	259	43	1	1
All Years	1315	1770	21	54	1558	391	1	1

Mean Score/Question/School Year

School Year	Number of Students	Q1-8	Q9	Q10	Overall Mean
8	573	8.7	0.3	0.1	9.0
9	724	9.9	0.4	0.2	10.5
10	725	10.9	0.5	0.4	11.7
Other	456	7.1	0.2	0.1	7.4
All Years	2478	9.4	0.4	0.2	9.9