The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 1995 Problems

1. Show that there are not more than 9 prime numbers bewteeen 10 and 10^{29} whose decimal representation is a string of 1s.

2. On the circumference of a circle, $4n$ points ($n \in \mathbb{N}$) have been chosen and numbered consecutively $1, 2, \ldots, 4n$ clockwise around the circle. The $2n$ even-numbered points are divided into n pairs, and the points of each pair joined by a green chord. Similarly, the $2n$ odd-numbered points are divided into n pairs, and the points of each pair joined by a gold chord. The choice of points and chords turns out to be such that no three of the chords (the same colour or a mixture of both) meet at a common point.

Prove that there are at least n points where a green chord intersects a gold chord.

3. A straight line cuts two concentric circles in the points A, B, C and D, in that order; AE and BF are parallel chords, one in each circle; $GC \perp BF$ at G, and $DH \perp AE$ at H.

Prove that $GF = HE$.

4. Determine all pairs (x, y) of positive integers which satisfy

$$y^2(x-1) = x^5 - 1.$$

5. Determine all $r \in \mathbb{R}$ such that there is precisely one pair (x, y) of real numbers satisfying the conditions

(i) $y - x = r$;

(ii) $x^2 + y^2 + 2x \leq 1$.

6. For each $n \in \mathbb{N}$, let $f(n,0), f(n,1), \ldots, f(n,2n) \in \mathbb{Z}$ such that

> the polynomial $f(n,0) + f(n,1)x + \cdots + f(n,2n)x^{2n}$ is identical with the polynomial $(x^2+x+1)^n$.

Prove that there are infinitely many values of n for which precisely three of the integers $f(n,k)$, $k = 0, 1, \ldots, 2n$ are odd.

7. The lines joining the three vertices of $\triangle ABC$ to a point in its plane cut the sides opposite the vertices A, B, C in the points K, L, M, respectively. A line through M parallel to KL cuts BC at V and AK at W.

Prove that $VM = MW$.

8. Determine all functions $f : \mathbb{R}_{>0} \to \mathbb{R}$ such that

(i) $f(xy) = f(x)f\left(\frac{3}{y}\right) + f(y)f\left(\frac{3}{x}\right)\ \forall x, y \in \mathbb{R}_{>0}$;

(ii) $f(1) = \frac{1}{2}$.