The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 1996 Problems

1. Let $ABCDE$ be a convex pentagon such that $BC = CD = DE$ and each diagonal of the pentagon is parallel to one of its sides.

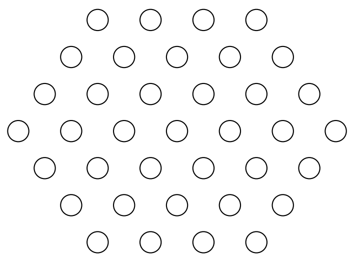
Prove that all the angles in the pentagon are equal and that all sides are equal.

2. Let $p(x)$ be acubic polynomial with roots r_1, r_2, r_3. Suppose that

$$\frac{p(\frac{1}{2}) + p(-\frac{1}{2})}{p(0)} = 1000.$$

Find the value of $\quad \dfrac{1}{r_1 r_2} + \dfrac{1}{r_2 r_3} + \dfrac{1}{r_3 r_1}$.

3. A number of tubes are bundled together in hexagonal form:



The number of tubes in the bundle can be $1, 7, 19, 37$ (as shown), $61, 91, \ldots$. If this sequence is continued, it will be noticed that the total number of tubes is often a number ending in 69.

What is the 69^{th} number of the sequence ending in 69?

4. For which $n \in \mathbb{N}$ can we rearrange the sequence $1, 2, \ldots, n$ to $a_1, a_2, \ldots, a_n$ such that

$$|a_k - k| = |a_1 - 1| \neq 0 \text{ for } k = 2, 3, \ldots, n?$$

5. Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $0 \leq s \in \mathbb{R}$ such that

(i) $a_1 \leq a_2 \leq \cdots \leq a_n$;

(ii) $a_1 + a_2 + \cdots + a_n = 0$;

(iii) $|a_1| + |a_2| + \cdots + |a_n| = s$.

Prove that $\quad a_n - a_1 \geq \dfrac{2s}{n}$.

6. Let $ABCD$ be a cyclic quadrilateral and let P and Q be points on the sides AB and AD, respectively, such that $AP = CD$ and $AQ = BC$. Let M be the point of intersection of AC and PQ.

Show that M is the midpoint of PQ.

7. For $n \in \mathbb{N}$, let $\sigma(n)$ denote the sum of all positive integers that divide n. Let $k \in \mathbb{N}$ and $n_1 < n_2, \ldots$ be an infinite sequence of poitive integer with the property that $\sigma(n_i) - n_i = k$ for $i = 1, 2, \ldots$.

Prove that n_i is prime for $i = 1, 2, \ldots$.

8. Let $f : \mathbb{Z} \to \{0, 1\}$ such that

(i) $f(n + 1996) = f(n) \ \forall n \in \mathbb{Z}$;

(ii) $f(1) + f(2) + \cdots + f(1996) = 45$.

Prove there exists $t \in \mathbb{Z}$ such that $f(n + t) = 0$ for all n for which $f(n) = 1$ holds.