The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 1997 Problems

1. Let ABC be a triangle with $AB = AC$ and $\angle BAC < 120^\circ$. Let D be the midpoint of BC. Choose point E on AD such that $\angle AEB = 120^\circ$. Let E' be any point on AD distinct from E. Prove that

$$EA + EB + EC < E'A + E'B + E'C.$$

2. Let $a_1, a_2, \ldots, a_k$ be real numbers satisfying the following two conditions:

(i) $0 \le a_1 \le a_2 \le \cdots \le a_k$;

(ii) $a_1 + a_2 + \cdots + a_k = 1$.

Prove that $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \le \dfrac{1}{k}$ for $n = 1, 2, \ldots, k$.

3. Determine all functions f defined for all real numbers and taking real number as values that satisfy the inequality

$$|f(x + h) - f(x)| \le h^2$$

for all real numbers x and h.

4. A staircase sequence is a sequence of ordered pairs of non-negative integers (x_1, y_1), (x_1, y_2), (x_2, y_2), (x_2, y_3), (x_3, y_3), (x_3, y_4), (x_4, y_4), (x_3, y_1), $\ldots$, in which $0 \le x_1 \le x_2 \le x_3 \le \cdots$ and $0 \le y_1 \le y_2 \le y_3 \le \cdots$. Prove that if each (x, y) in the coordinate plane, with x and y being non-negative integers, is coloured either red or blue, then it is possible to find an infinite staircase such that all its points are the same colour.

5. For each positive integer n let $p(n)$ be the product of positive integers that divide n. Prove that if a and b are positive integers and $p(a) = p(b)$, then $a = b$.

6. For any real number x, let $\lfloor x \rfloor$ denote the largest integer not exceeding x. Prove that if n is a positive integer, then

$$\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{4n+1} \right\rfloor.$$

7. Let m and n be integers greater than 1. Prove that

$$\frac{1}{\sqrt[m]{n+1}} + \frac{1}{\sqrt[n]{m+1}} > 1.$$

8. Let ABC be a triangle with $\angle ABC = 60^\circ$ and $\angle BAC = 40^\circ$. Let P be a point on AB such that $\angle BCP = 70^\circ$ and let Q be a point on AC such that $\angle CBQ = 40^\circ$. Let BQ intersect CP at R. Prove that AR (extended) is perpendicular to BC.