The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

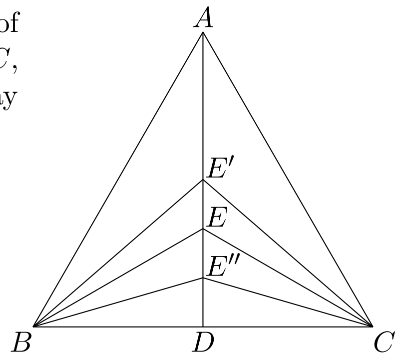
AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 1997 Problems with some Solutions

1. Let ABC be a triangle with $AB = AC$ and $\angle BAC < 120^\circ$. Let D be the midpoint of BC. Choose point E on AD such that $\angle AEB = 120^\circ$. Let E' be any point on AD distinct from E. Prove that

$$EA + EB + EC < E'A + E'B + E'C.$$

Solution.
Since $AB = AC$, $\triangle ABC$ is isosceles and so AD is an axis of symmetry for $\triangle ABC$. In particular, $EB = EC$ and $E'B = E'C$, so that the given inequality that we are required to prove may be rewritten as

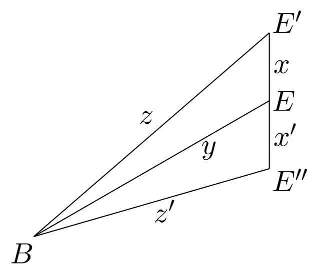


$$EA + 2EB < E'A + 2E'B$$

or with a little rearrangement and division by 2 as

$$E'B > EB + \tfrac{1}{2}(EA - E'A).$$

Let us write $z = E'B$, $y = EB$ and $x = EE'$. Then, if E' lies between E and A (Case 1., below), we must show



$$z > y + \tfrac{1}{2}x,$$

and, if E' lies between E and D (Case 2.) we must show

$$z > y - \tfrac{1}{2}x.$$

In the diagram, we have labelled the location of E' for Case 2. as E''.

Case 1: E' lies between E and A. Applying the Cosine Rule to $\triangle EBE'$, noting that $\angle BEE' = \angle BEA = 120^\circ$ we have

$$\begin{aligned} z^2 &= y^2 + x^2 - 2yx\cos 120^\circ \\ &= y^2 + x^2 - 2yx \cdot -\tfrac{1}{2} \\ &= y^2 + x^2 + yx \\ &> y^2 + \tfrac{1}{4}x^2 + 2 \cdot \tfrac{1}{2}\,xy \\ &= (y + \tfrac{1}{2}\,x)^2 \\ \therefore z &> y + \tfrac{1}{2}\,x. \end{aligned}$$

Thus we have established what was required in this case.

Case 2: E' lies between E and D. (In the diagram, E', x and z are represented by E'', x' and z', respectively, for this case.) Applying the Cosine Rule to $\triangle EBE''$, noting that $\angle BEE'' = 60°$, we have

$$\begin{aligned} z^2 &= y^2 + x^2 - 2yx\cos 60° \\ &= y^2 + x^2 - 2yx \cdot \tfrac{1}{2} \\ &= y^2 + x^2 - yx \\ &> y^2 + \tfrac{1}{4}x^2 - 2\cdot\tfrac{1}{2}\,xy \\ &= (y - \tfrac{1}{2}\,x)^2 \\ \therefore z &> |y - \tfrac{1}{2}\,x| \geq y - \tfrac{1}{2}\,x. \end{aligned}$$

Thus we have also established what was required in this case.

2. Let $a_1, a_2, \dots, a_k$ be real numbers satisfying the following two conditions:

(i) $0 \leq a_1 \leq a_2 \leq \cdots \leq a_k$;

(ii) $a_1 + a_2 + \cdots + a_k = 1$.

Prove that $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \leq \dfrac{1}{k}$ for $n = 1, 2, \dots, k$.

Solution. Firstly,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \frac{a_n + a_n + \cdots + a_n}{n} = a_n \tag{1}$$

Hence, for $1 \leq n < k$,

$$\begin{aligned} \frac{n+1}{n}(a_1 + a_2 + \cdots + a_n) &= a_1 + a_2 + \cdots + a_n + \frac{a_1 + a_2 + \cdots + a_n}{n} \\ &\leq a_1 + a_2 + \cdots + a_n + a_n, && \text{by (1)} \\ &\leq a_1 + a_2 + \cdots + a_n + a_{n+1}, && \text{since } a_n \leq a_{n+1} \text{ by (i)} \\ \therefore \frac{a_1 + a_2 + \cdots + a_n}{n} &\leq \frac{a_1 + a_2 + \cdots + a_n + a_{n+1}}{n+1} && (2) \end{aligned}$$

Let $b_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$ for $n = 1, 2, \dots, k$. Then (2) shows that

$$b_n \leq b_{n+1},$$

for $1 \leq n < k$, which is to say that b_n is a non-decreasing sequence, i.e.

$$a_1 = \frac{a_1}{1} = b_1 \leq b_2 \leq \cdots \leq b_k = \frac{a_1 + a_2 + \cdots + a_k}{k} = \frac{1}{k},$$

using (ii). In particular,

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \leq \frac{1}{k},$$

for $n = 1, 2, \dots, k$, as required.

3. Determine all functions f defined for all real numbers and taking real number as values that satisfy the inequality

$$|f(x+h)-f(x)| \leq h^2$$

for all real numbers x and h.

Solution. Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then

$$f(0)-f(x) = f(0) - f\Big(\frac{x}{n}\Big) + f\Big(\frac{x}{n}\Big) - f\Big(\frac{2x}{n}\Big) + \cdots + f\Big(\frac{(n-1)x}{n}\Big) - f(x).$$

Thus, by the Triangle Inequality,

$$\begin{aligned} |f(0)-f(x)| &\leq \Big|f(0) - f\Big(\frac{x}{n}\Big)\Big| + \Big|f\Big(\frac{x}{n}\Big) - f\Big(\frac{2x}{n}\Big)\Big| + \cdots + \Big|f\Big(\frac{(n-1)x}{n}\Big) - f(x)\Big| \\ &\leq \Big(\frac{x}{n}\Big)^2 + \Big(\frac{x}{n}\Big)^2 + \cdots + \Big(\frac{x}{n}\Big)^2, \quad \text{using the given inequality with } h = x/n \\ &= n\Big(\frac{x}{n}\Big)^2 = \frac{x^2}{n}. \end{aligned}$$

Since the above is true for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we must have $|f(0)-f(x)|$. Hence $f(0) = f(x)$ for all $x \in \mathbb{R}$, i.e. if $f(x)$ satisfies the given condition then it is constant.
Now, suppose $f(x) = c$ (constant), for all $x \in \mathbb{R}$. Then $|f(x+h)-f(x)| = |c-c| = 0 \leq h^2$. Thus the constant functions do indeed satisfy the condition.
Therefore the set of all constant functions is the required set of functions that satisfy

$$|f(x+h)-f(x)| \leq h^2$$

for all real numbers x and h.

In calculus, the Squeeze Theorem states:

> If $g(x) \leq f(x) \leq k(x)$ in a neighbourhood of $x = a$ and $\lim_{x\to a} g(x) = L = \lim_{x\to a} k(x)$ for some $L \in \mathbb{R}$ then $\lim_{x\to a} f(x) = L$.

Thus for $h \neq 0$, we have

$$\begin{aligned} |f(x+h)-f(x)| &\leq h^2 = |h|^2 \\ \Big|\frac{f(x+h)-f(x)}{h}\Big| &\leq |h| \\ |f'(x)| = \lim_{h\to 0}\Big|\frac{f(x+h)-f(x)}{h}\Big| &\leq \lim_{h\to 0}|h| = 0 \end{aligned}$$

and hence $f'(x) = 0$ for all $x \in \mathbb{R}$, from which it follows that f is constant.

4. A staircase sequence is a sequence of ordered pairs of non-negative integers (x_1, y_1), (x_1, y_2), (x_2, y_2), (x_2, y_3), (x_3, y_3), (x_3, y_4), (x_4, y_4), (x_3, y_1), $\ldots$, in which $0 \le x_1 \le x_2 \le x_3 \le \cdots$ and $0 \le y_1 \le y_2 \le y_3 \le \cdots$. Prove that if each (x, y) in the coordinate plane, with x and y being non-negative integers, is coloured either red or blue, then it is possible to find an infinite staircase such that all its points are the same colour.

Solution. If there are infinitely many horizontal lines, each of which contains only a finite number of blue points, then we can form a staircase as follows. Suppose that the horizontal lines are $y = y_1$, $y = y_2$, $y = y_2$ .... Take a sufficiently large integer x_1 such that for all $x \ge x_1$, the points (x, y_1) and (x, y_2) are red; then take x_2 such that for all $x \ge x_2$, the point (x, y_3) is red and so on. Our staircase is then (x_1, y_1), (x_1, y_2), (x_2, y_2), (x_2, y_3), (x_3, y_3) ....
The same argument applies if there are infinitely many horizontal lines, each of which contains only a finite number of red points, or if there are infinitely many vertical lines which contain only a finite number of points of one colour.
If none of these conditions applies, then there is a point (x_1, y_1) such that all vertical lines $x = a$ and all horizontal lines $y = b$ for $a \ge x_1$ and $b \ge y_1$ contain infinitely points of either colour. We can the form a staircase by starting with (x_1, y_1), then moving up till we find a point of the same colour, then right till we find a point of the same colour, and so on.

5. For each positive integer n let $p(n)$ be the product of positive integers that divide n. Prove that if a and b are positive integers and $p(a) = p(b)$, then $a = b$.

6. For any real number x, let $\lfloor x \rfloor$ denote the largest integer not exceeding x. Prove that if n is a positive integer, then

$$\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{4n+1} \right\rfloor.$$

Solution. Let $n \in \mathbb{N}$.

$$\begin{aligned}(\sqrt{n} + \sqrt{n+1})^2 &= n + n + 1 + 2\sqrt{n(n+1)} \\ &= 2n + 1 + 2\sqrt{n^2 + n}\end{aligned}$$

Now

$$\begin{aligned}n^2 < n^2 + n & \quad < n^2 + 2n + 1 = (n+1)^2 \\ \therefore n < \sqrt{n^2 + n} & \quad < n + 1 \\ \therefore 4n + 1 = 2n + 1 + 2n < (\sqrt{n} + \sqrt{n+1})^2 & \quad < 2n + 1 + 2(n+1) = 4n + 3 \\ \therefore \sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} & \quad < \sqrt{4n+3} \\ \therefore \lfloor \sqrt{4n+1} \rfloor \le \lfloor \sqrt{n} + \sqrt{n+1} \rfloor & \quad \le \lfloor \sqrt{4n+3} \rfloor \qquad (3)\end{aligned}$$

An integer square is only of form $4N$ or $4N + 1$, $N \in \mathbb{N} \cup \{0\}$ (see the dangerous bend for a proof).

Lemma. An integer square is only of form $4N$ or $4N + 1$, $N \in \mathbb{N} \cup \{0\}$.

Proof. Let $k \in \mathbb{N} \cup \{0\}$. Then $k \equiv 0, 1, 2, 3 \pmod 4$, and hence $k^2 \equiv 0, 1 \pmod 4$, since $4 \equiv 0 \pmod 4$ and $9 \equiv 1 \pmod 4$. □

Thus, it follows that

$$\lfloor \sqrt{4n+1} \rfloor = \lfloor \sqrt{4n+3} \rfloor$$

and so the lower and upper bounds of $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor$ in (3) are equal, and hence

$$\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor$$

7. Let m and n be integers greater than 1. Prove that

$$\frac{1}{\sqrt[m]{n+1}} + \frac{1}{\sqrt[n]{m+1}} > 1.$$

Solution. By the Binomial Theorem, for $x > 0$ and $2 \le k \in \mathbb{N}$,

$$\begin{aligned}(1+x)^k &= 1 + kx + \binom{k}{2}x^2 + \cdots \\ &> 1 + kx\end{aligned}$$

and so putting $x = n/m$, where $2 \le m, n \in \mathbb{N}$, we have

$$\begin{aligned}\left(1 + \frac{n}{m}\right)^m &> 1 + m \cdot \frac{n}{m} = 1 + n \\ \therefore 1 + \frac{n}{m} &> \sqrt[m]{1+n} \\ \therefore \frac{1}{\sqrt[m]{1+n}} &> \frac{1}{1 + \frac{n}{m}} = \frac{n}{m+n}\end{aligned}$$

Similarly, (interchanging m and n),

$$\begin{aligned}\frac{1}{\sqrt[n]{1+m}} &> \frac{1}{1 + \frac{m}{n}} = \frac{m}{n+m} \\ \therefore \frac{1}{\sqrt[m]{1+n}} + \frac{1}{\sqrt[n]{1+m}} &> \frac{n}{m+n} + \frac{m}{n+m} = 1\end{aligned}$$

8. Let ABC be a triangle with $\angle ABC = 60°$ and $\angle BAC = 40°$. Let P be a point on AB such that $\angle BCP = 70°$ and let Q be a point on AC such that $\angle CBQ = 40°$. Let BQ intersect CP at R. Prove that AR (extended) is perpendicular to BC.