The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 1998 Problems

1. Determine all real solutions of the equation

$$(x+1998)(x+1999)(x+2000)(x+2001)+1=0.$$

2. Find all pairs (r,s) of non-negative real numbers that satisfy the following two conditions:

(i) $2^{r^4+s^2}+2^{r^2+s^4}=8$;

(ii) $r+s=2$.

3. In $\triangle ABC$, let D be a point on AB and E be a point on AC such that $DE \parallel BC$ and DE is tangent to the incircle of $\triangle ABC$.

Prove that $8DE \leq AB+BC+CA$.

4. Determine all functions $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ such that

(i) $f(-x)=f(x)$ for all $0\neq x\in\mathbb{R}$;

(ii) $f\Big(\dfrac{1}{x+y}\Big)=f\Big(\dfrac{1}{x}\Big)+f\Big(\dfrac{1}{y}\Big)+2(xy-1000)$ for all $x,y\in\mathbb{R}$ such that $x,y,x+y\neq 0$.

5. Consider all binary 2×2 arrays, where *binary* means each entry is 0 or 1. We say that a pair A,B of such 2×2 arrays is *compatible* if there exists a 3×3 array within which both A and B appear as 2×2 arrays, e.g., the pair

$$\begin{pmatrix}1&0\\1&1\end{pmatrix},\quad \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

are compatible since both arrays occur within

$$\begin{pmatrix}1&0&0\\1&1&0\\1&0&0\end{pmatrix}.$$

Determine all pairs of binary 2×2 arrays that are not compatible.

6. Prove that for all $n\in\mathbb{N}$,

$$(1998n)!\leq\Big(\frac{3995n+1}{2}\cdot\frac{3993n+1}{2}\cdot\frac{3991n+1}{2}\cdot\cdots\cdot\frac{n+1}{2}\Big)^n.$$

7. Suppose $\triangle ABC$ has area $1998\,\text{cm}^2$. Let G be the centroid of $\triangle ABC$. Each line through G cuts $\triangle ABC$ into two regions having areas A_1 and A_2, say.

Determine the largest possible value of A_1-A_2.

8. A team of archaeologists was able to establish that 40 000 years ago, there had been a flourishing civilisation in the crocodile-infested jungle of Udakak. As many as 12 cities were excavated. After some effort, the Udakak script was deciphered, and the rich Udakak literature was studied by historians. They discovered that

(i) Udakak could not have had more than 28 cities;

(ii) the Udakakan cities had peculiar trade arrangements:

  (a) whenever a city A had no trade relations with another city B, there were exactly two cities with which A and B both had trade relations;

  (b) whenever a city A had trade relations with another city B, no city had trade relations with both A and B.

For budgetary reasons, the archaeologists wanted to know how much excavation work was still ahead of them.

Show how the archaeologists can work out the exact number of Udakakan cities.