

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 1998 Problems
with Solution to Problem 8

1. Determine all real solutions of the equation

$$(x + 1998)(x + 1999)(x + 2000)(x + 2001) + 1 = 0.$$

2. Find all pairs (r, s) of non-negative real numbers that satisfy the following two conditions:

(i) $2^{r^4+s^2} + 2^{r^2+s^4} = 8;$

(ii) $r + s = 2.$

3. In $\triangle ABC$, let D be a point on AB and E be a point on AC such that $DE \parallel BC$ and DE is tangent to the incircle of $\triangle ABC$.

Prove that $8DE \leq AB + BC + CA$.

4. Determine all functions $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ such that

(i) $f(-x) = f(x)$ for all $0 \neq x \in \mathbb{R};$

(ii) $f\left(\frac{1}{x+y}\right) = f\left(\frac{1}{x}\right) + f\left(\frac{1}{y}\right) + 2(xy - 1000)$ for all $x, y \in \mathbb{R}$ such that $x, y, x + y \neq 0$.

5. Consider all binary 2×2 arrays, where *binary* means each entry is 0 or 1. We say that a pair A, B of such 2×2 arrays is *compatible* if there exists a 3×3 array within which both A and B appear as 2×2 arrays, e.g., the pair

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are compatible since both arrays occur within

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Determine all pairs of binary 2×2 arrays that are not compatible.

6. Prove that for all $n \in \mathbb{N}$,

$$(1998n)! \leq \left(\frac{3995n+1}{2} \cdot \frac{3993n+1}{2} \cdot \frac{3991n+1}{2} \cdots \frac{n+1}{2} \right)^n.$$

7. Suppose $\triangle ABC$ has area 1998 cm^2 . Let G be the centroid of $\triangle ABC$. Each line through G cuts $\triangle ABC$ into two regions having areas A_1 and A_2 , say.

Determine the largest possible value of $A_1 - A_2$.

8. A team of archaeologists was able to establish that 40 000 years ago, there had been a flourishing civilisation in the crocodile-infested jungle of Udakak. As many as 12 cities were excavated. After some effort, the Udakak script was deciphered, and the rich Udakak literature was studied by historians. They discovered that

- (i) Udakak could not have had more than 28 cities;
- (ii) the Udakakan cities had peculiar trade arrangements:
 - (a) whenever a city A had no trade relations with another city B , there were exactly two cities with which A and B both had trade relations;
 - (b) whenever a city A had trade relations with another city B , no city had trade relations with both A and B .

For budgetary reasons, the archaeologists wanted to know how much excavation work was still ahead of them.

Show how the archaeologists can work out the exact number of Udakakan cities.

Solution. We have the number n of Udakakan cities satisfies $12 \leq n \leq 28$, together with the conditions:

- (a) If city A did not trade with city B , then there were exactly two cities with which A and B both traded.
- (b) If city A did trade city B , then no city traded with both A and B .

Let X be one of the Udakakan cities. Let N_1, \dots, N_k be the non-trader cities with X , and let T_1, \dots, T_m be the trader cities with X , so that the total number of cities is $1 + k + m$.

Consider a pair T_i, T_j . Being trader cities with X , they are non-traders with each other, otherwise X and T_i would have both traded with T_j , contradicting (b). However, by (a), T_i, T_j have exactly one other mutual trading partner besides X , which must therefore be an N_ℓ . On the other hand, each N_i has exactly two trading partners among the T_j s, by (a). These two last statements together imply $k = \binom{m}{2}$. Hence, $n = 1 + m + \binom{m}{2}$, and

$$1 + 4 + \binom{4}{2} = 11 < 12 < 16 = 1 + 5 + \binom{5}{2} < 1 + 6 + \binom{6}{2} = 22 < 28 < 29 = 1 + 7 + \binom{7}{2}.$$

Therefore, $m = 5$ (and $n = 16$) or $m = 6$ (and $n = 22$).

Let's label the common trading partner of T_i, T_j , besides X , as N_{ij} , $i < j$.

Suppose first that $m = 6$. Since any city can be chosen to have the role of X , every city had exactly 6 trading partners. Two of the trading partners of N_{12} are T_1 and T_2 . The other four trading partners must be among the N s, and in particular, $N_{34}, N_{35}, N_{36}, N_{45}, N_{46}, N_{56}$. Without loss of generality, N_{12} does not trade with N_{34} , say, but then by (a), N_{12} and N_{34} ought to have two mutual trading partners, but can have at most one: N_{56} .

Thus, we are left only with one possibility, $m = 5$ (and $n = 16$), which is possible if, for example, exactly the following pairs of cities were trading partners:

$$\begin{aligned} &(X, T_i), i = 1, 2, \dots, 5; \\ &(T_i, N_{ij}), (T_j, N_{ij}), 1 \leq i < j \leq 5; \\ &(N_{ij}, N_{k\ell}), 1 \leq i < j \leq 5, 1 \leq k < \ell \leq 5 \text{ and } \{i, j\} \cap \{k, \ell\} = \emptyset. \end{aligned}$$

Thus the archaeologists need to seek funding for exactly 16 cities.