The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 1999 Problems

1. Points P, Q, R, S lie, in that order, on a circle such that $PQ \parallel SR$ and $QR = SR$. Point T lies in the same plane as the circle such that QT is tangent to the circle and $\angle RQT$ is acute.

Prove that

(a) $PS = QR$;

(b) $\angle PQT$ is trisected by QR and QS.

2. A town has 99 clubs $C_1, C_2, \dots, C_{99}$, each of which has at least one member, and no two of which have exactly the same members.

Determine the least $n \in \mathbb{N}$ such that one can be certain there is a set S of n people with the property:

> whenever C_i and C_j, $1 \le i, j \le 99$, are different clubs in the town, then either there is a person in S who belongs to C_i but not to C_j, or there is a person in S who belongs to C_j but not to C_i.

3. (a) Find $a_1, a_2, a_3, d_3 \in \mathbb{N}$ such that

(i) $a_k - a_{k-1} = d_3$ for $k = 2, 3$, and

(ii) there are $m_i, b_i \in \mathbb{Z}$ such that $m_i > 1$ and $a_i = b_i^{m_i}$ for $i = 1, 2, 3$.

(b) Show for each integer $n > 1$, there exist $a_1, a_2, \dots, a_n, d_n \in \mathbb{N}$ such that

(i) $a_k - a_{k-1} = d_n$ for $k = 2, 3, \dots, n$, and

(ii) there are $m_i, b_i \in \mathbb{Z}$ such that $m_i > 1$ and $a_i = b_i^{m_i}$ for $i = 1, 2, \dots, n$.

4. In triangle $\triangle$, the radius of the incircle is r.

Prove that the sum of the lengths of the altitudes of $\triangle$ is at least $9r$.

5. Let $1 < x \in \mathbb{R}$ and $1 < n \in \mathbb{Z}$. Prove that

$$1 + \frac{x-1}{nx} < \sqrt[n]{x} < 1 + \frac{x-1}{n}.$$

6. For $\triangle ABC$, points D, E, F are in its exterior such that $\triangle ABD$, $\triangle BCE$, $\triangle CAF$ are equilateral. The sides of these triangles are extended so that BE and AF meet at K, DB and FC meet at L, and DA and EC meet at M.

Prove that $DK \parallel EL \parallel FM$.

7. Let $n \in \mathbb{Z}$ and p be a prime such that $1 + np$ is a perfect square.

Prove that $n + 1$ is the sum of p perfect squares.

8. (a) Find an integer sequence $a_1, a_2, \dots$ with the properties

(i) $a_n \in \{1, -1\}$ for $n \in \mathbb{N}$;

(ii) $a_{mn} = a_m a_n$ for all $m, n \in \mathbb{N}$;

(iii) for no $n \in \mathbb{N}$, does $a_n = a_{n+1} = a_{n+2}$ hold.

(b) Determine all integer sequence $a_1, a_2, \dots$ with the properties (i), (ii), (iii).