The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 1999 Problems with Some Solutions and Some Hints

1. Points P, Q, R, S lie, in that order, on a circle such that $PQ \parallel SR$ and $QR = SR$. Point T lies in the same plane as the circle such that QT is tangent to the circle and $\angle RQT$ is acute.

Prove that

(a) $PS = QR$;

(b) $\angle PQT$ is trisected by QR and QS.

2. A town has 99 clubs $C_1, C_2, \dots, C_{99}$, each of which has at least one member, and no two of which have exactly the same members.

Determine the least $n \in \mathbb{N}$ such that one can be certain there is a set S of n people with the property:

> whenever C_i and C_j, $1 \le i, j \le 99$, are different clubs in the town, then either there is a person in S who belongs to C_i but not to C_j, or there is a person in S who belongs to C_j but not to C_i.

Solution. The answer is $|S| = n = 98$. We will show by an example that $n \ge 98$ and then show $n \le 98$ by strong induction on the number of clubs in the town.
First, consider a town with 99 people, each in their own club with no other members. If there were a set S with $|S| = 97$, then a least two clubs C_1, C_2 would have no members in S. Hence, S has no member in C_1, which is not in C_2, and vice-versa. $\therefore n \ge 98$.
Next, consider a town with exactly 2 clubs C_1, C_2, which satisfy the given conditions, i.e. $C_1 \ne C_2$. Now, since $C_1 \ne C_2$, the clubs have at least one member x not in both of them. Let $S = \{x\}$; then $|S| = 1 = 2 - 1$ and S satisfies the condition: either x is in C_1, but not in C_2, or vice-versa. (Note, if the town has one club, we may take S to be the empty set, and $|S| = 0 = 1 - 1$.)
We now assume that if a town has 1 or 2 or $\cdots$ or$k$ clubs, with $C_i \ne C_j$ for all $i \ne j$, then we can find a set S with $0, 1, 2, \dots, k-1$ members satisfying "the condition": given $C_i \ne C_j$ there is an x in C_i but not C_j, or vice-versa.
Now consider a town with $k+1$ clubs, $C_1, C_2, \dots, C_{k+1}$, satisfying $C_i \ne C_j$ for all $i \ne j$. Since $C_1 \ne C_2$, there is a person, call him Fred, who is in C_1, but not C_2, or who is in C_2, but not C_1. Let X be the set of clubs Fred is in and X', the set of clubs Fred is not in; $|X| \ge 1$, since Fred is in at least one club, and $|X'| \ge 1$, since Fred is not in at least one club. But $|X| + |X'| = k+1$; hence $|X| \le k$ and $|X'| \le k$.
Since $|X| \le k$, there is a set S_X with $\le k-1$ members satisfying "the condition" in the "town" with clubs in X only. Similarly, there is a set $S_{X'}$ with $|X'| - 1$ members satisfying "the condition" for clubs in X'. Finally, consider $S_X \cup S_{X'} \cup \{\text{Fred}\}$.

$$\begin{aligned} |S_X| &\le |S_X| + |S_{X'}| + 1 \\ &\le |X| - 1 + |X'| - 1 + 1 \\ &= |X| + |X'| - 1 \\ &= k + 1 - 1 = k. \end{aligned}$$

Consider two clubs C_i, C_j. If $C_i \in X$ and $C_j \in X'$ then Fred is in C_i but C_j. If $C_i, C_j \in X'$ then there is $x \in S_{X'}$ with the required property.
Thus the induction is complete. So, if $k + 1 = 99$, $|S| = n \le 98$.

3. (a) Find $a_1, a_2, a_3, d_3 \in \mathbb{N}$ such that

(i) $a_k - a_{k-1} = d_3$ for $k = 2, 3$, and

(ii) there are $m_i, b_i \in \mathbb{Z}$ such that $m_i > 1$ and $a_i = b_i^{m_i}$ for $i = 1, 2, 3$.

(b) Show for each integer $n > 1$, there exist $a_1, a_2, \ldots, a_n, d_n \in \mathbb{N}$ such that

(i) $a_k - a_{k-1} = d_n$ for $k = 2, 3, \ldots, n$, and

(ii) there are $m_i, b_i \in \mathbb{Z}$ such that $m_i > 1$ and $a_i = b_i^{m_i}$ for $i = 1, 2, \ldots, n$.

Solution.

(a) There are many possible solutions: $a_1 = 1, a_2 = 25, a_3 = 49, d_3 = 24$ is the most obvious.

(b) We induct on n. For $n = 2$, take $a_1 = 1$, $a_2 = 4$, $d_2 = 3$.
Now let $n > 2$ and assume there are $c_1, c_2, \ldots, c_{n-1}, e \in \mathbb{N}$ such that

$$c_k - c_{k-1} = e \text{ for } k = 2, 3, \ldots, n-1$$

and such that

$$c_i = f_i^{m_i} \text{ for } i = 1, 2, \ldots, n-1$$

for some integers $f_i, m_i > 1$. Let

$$m = \operatorname{lcm}(m_1, m_2, \ldots, m_{n-1}) \text{ and } c = c_{n-1} + e.$$

Put

$$a_i = c_i c^m, \text{ for } i = 1, 2, \ldots, n-1$$

and $a_n = c^{m+1}$. Then

$$a_i = f_i^{m_i} c^m = (f_i d)^{m_i} \text{ where } d^{m_i} = c^m.$$

Also,

$$a_{i+1} - a_i = f_{i+1}^{m_{i+1}} c^m - f_i^{m_i} c^m = ec^m$$

and

$$a_{n+1} - a_n = c^{m+1} - c_{n-1} c^m = (c - c_{n-1}) c^m = ec^m.$$

So, $a_1, a_2, \ldots, a_n$ and $d_n = c^m e$ are as required.

4. In triangle $\triangle$, the radius of the incircle is r.

Prove that the sum of the lengths of the altitudes of $\triangle$ is at least $9r$.

5. Let $1 < x \in \mathbb{R}$ and $1 < n \in \mathbb{Z}$. Prove that

$$1 + \frac{x-1}{nx} < \sqrt[n]{x} < 1 + \frac{x-1}{n}.$$

6. For $\triangle ABC$, points D, E, F are in its exterior such that $\triangle ABD$, $\triangle BCE$, $\triangle CAF$ are equilateral. The sides of these triangles are extended so that BE and AF meet at K, DB and FC meet at L, and DA and EC meet at M.

Prove that $DK \parallel EL \parallel FM$.

7. Let $n \in \mathbb{Z}$ and p be a prime such that $1 + np$ is a perfect square.

Prove that $n + 1$ is the sum of p perfect squares.

8. (a) Find an integer sequence $a_1, a_2, \ldots$ with the properties

(i) $a_n \in \{1, -1\}$ for $n \in \mathbb{N}$;

(ii) $a_{mn} = a_m a_n$ for all $m, n \in \mathbb{N}$;

(iii) for no $n \in \mathbb{N}$, does $a_n = a_{n+1} = a_{n+2}$ hold.

(b) Determine all integer sequence $a_1, a_2, \ldots$ with the properties (i), (ii), (iii).

Solution. We claim that there are exactly two sequences satisfying (i), (ii), (iii), either of which solves (a). Let the two sequences be $\left(b_n\right)_{n=1}^{\infty}$ and $\left(c_n\right)_{n=1}^{\infty}$ defined by

$$b_n = \begin{cases} 1, & \text{if } n \equiv 1 \pmod 3 \\ -1, & \text{if } n \equiv -1 \pmod 3 \\ 1, & \text{if } n = 3 \\ b_q, & \text{if } n = 3^k q \text{ for some } k \text{ s.t. } 3 \nmid q \end{cases} \quad \text{and } c_n = \begin{cases} 1, & \text{if } n \equiv 1 \pmod 3 \\ -1, & \text{if } n \equiv -1 \pmod 3 \\ -1, & \text{if } n = 3 \\ (-1)^k c_q, & \text{if } n = 3^k q \text{ for some } k \text{ s.t. } 3 \nmid q. \end{cases}$$

Observe that (i) is satisfied by both sequences and since each sequence is: $1, -1, \pm 1, 1, -1, \pm 1, \ldots,$ no three consecutive elements are equal, and so (iii) is satisfied.
Checking (ii) takes a little more care:

First observe that if $n \not\equiv 0 \pmod 3$ then the definition of b_n shows that $b_n \equiv n \pmod 3$. Thus, if $m, n \not\equiv 0 \pmod 3$ then $mn \not\equiv 0 \pmod 3$, so that

$$b_{mn} \equiv mn \equiv b_m b_n \pmod 3,$$

and each of $b_{mn}, b_m, b_n \in \{1, -1\}$. Hence,

$$b_{mn} = b_m b_n,$$

in this case, and since $b_n = c_n$ under the same conditions, the same is true for the c_n sequence.
There are two more combinations to check.

Proving there are only two sequences takes some effort, using complete induction to prove that any sequence $\left(a_n\right)_{n=1}^{\infty}$ satisfying (i), (ii), (iii), satisfies

$$a_n = \begin{cases} 1, & \text{if } n \equiv 1 \pmod 3 \\ -1, & \text{if } n \equiv -1 \pmod 3 \end{cases} \tag{1}$$

(This is sufficient, since after defining a_3 as 1 or -1. condition (ii) determines all other terms.)
First $a_1 = (a_1)^2 = 1$. Now, suppose for a contradiction that $a_2 \neq -1$. Then $a_2 = 1$ and $a_3 = -1$ (since otherwise $a_1 = a_2 = a_3$ contravening (iii)). Following this through, we eventually find a contradiction with $a_8 = a_9 = a_{10} = 1$ (by contravening (iii)). Thus $a_2 = -1$.
Next assume $a_{3k+1} = 1$, $a_{3k+2} = -1$, $1 \le k \le n-1$. There are 4 cases for (a_{3n+1}, a_{3n+2}):
Case 1: $a_{3n+1} = a_{3n+2} = 1$. Eventually, we get a contradiction with

$$a_{2n} = a_{2n+1} = a_{2n+2} = \frac{1}{a_3}.$$

Case 2: $a_{3n+1} = a_{3n+2} = -1$. This leads to a contradiction similar to Case 1.
Case 3: $a_{3n+1} = -1, a_{3n+2} = 1$. This leads to a contradiction with (ii) being contravened.
Case 4: $a_{3n+1} = 1, a_{3n+2} = -1$. This being the last remaining case it must occur, and the induction is complete. So (1) holds for all $n \in \mathbb{N}$ and the sequences $\left(b_n\right)_{n=1}^{\infty}$ and $\left(c_n\right)_{n=1}^{\infty}$ are the only ones satisfying (i), (ii), (iii).