

The University of Western Australia  
SCHOOL OF MATHEMATICS & STATISTICS  
AMO TRAINING SESSIONS

**Australian Mathematics Olympiad, 2000 Problems**

1. Find all polynomials  $f$  with real coefficients such that

$$(x - 27)f(3x) = 27(x - 1)f(x) \quad \text{for all } x \in \mathbb{R}.$$

2. For each date of the year 2000, evaluate the expression

$$\text{day}^{\text{month}} - \text{year}$$

and then find the highest power of 5 dividing it, e.g. for 15 April, 2000, we obtain

$$\text{day}^{\text{month}} - \text{year} = 15^4 - 2000 = 5^3(405 - 16) = 5^3 \cdot 389,$$

for which the highest power of 5 dividing it, is  $5^3$ .

Find all dates for which the corresponding power of 5 is the greatest.

3. Let  $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in \mathbb{R}$  such that

(i)  $0 < x_1y_1 < x_2y_2 < \dots < x_ny_n$  and

(ii)  $x_1 + x_2 + \dots + x_i \geq y_1 + y_2 + \dots + y_i$  for  $1 \leq i \leq n$ .

Prove that 
$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \leq \frac{1}{y_1} + \frac{1}{y_2} + \dots + \frac{1}{y_n}.$$

When does equality occur?

4. Let  $A, B, C, A', B', C'$  be points on a circle such that  $AA' \perp BC$ ,  $BB' \perp CA$ ,  $CC' \perp AB$ . Further, let  $D$  be on the circle, and let  $DA'$  meet  $BC$  at  $A''$ ,  $DB'$  meet  $CA$  at  $B''$ ,  $DC'$  meet  $AB$  at  $C''$ , all line segments being extended where required.

Prove that  $A'', B'', C''$  and the orthocentre of  $\triangle ABC$  are collinear.

5. Let  $m, n \in \mathbb{N}$ . Prove that  $m^{n+2} + m^{n+1} + m^n$  is not a perfect square.

6. Let  $a, b, c \in \mathbb{R}$ , not all zero. Determine all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$af(xy + z^2) + bf(yz + x^2) + cf(zx + y^2) = 0 \quad \text{for all } x, y, z \in \mathbb{R}.$$

Distinguish all possibilities for  $a, b, c$ .

7. Solve the following system of equations

$$x + \lfloor y \rfloor + \{z\} = 200.0$$

$$\{x\} + y + \lfloor z \rfloor = 190.1$$

$$\lfloor x \rfloor + \{y\} + z = 178.8,$$

where for  $r \in \mathbb{R}$ ,  $\{r\} = r - \lfloor r \rfloor$ , the “fractional part” of  $r$ .

8. Let  $A_1, A_2, A_3, A_4, A_5, A_6, A_7$  be vertices of a heptagon lying in plane  $\Pi$ , and let  $B$  and  $C$  be different points not lying in  $\Pi$  and such that no three of these nine points are collinear. Now colour each of the 14 edges  $A_iB$  and  $A_iC$  ( $i = 1, 2, \dots, 7$ ), the 14 diagonals of  $A_1A_2A_3A_4A_5A_6A_7$ , and the line segment  $BC$ , either green or gold.

Prove that there are three line segments of the same colour that form a triangle.