The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2001 Problems

1. Let $L(n)$ be the least common multiple of $1, 2, \ldots, n$.

Determine all prime number pairs (p, q) such that $q = p + 2$ and $L(q) > qL(p)$.

2. Let $\triangle BCA$ be isosceles with $BC = CA$, and circumcentre O. Let P, Q, R be points on AB, BC, CA, respectively, such that $PQ \parallel AC$ and $PR \parallel BC$.

Prove that the quadrilateral $CROQ$ is cyclic.

3. A town has c celebrities and m magazines. One week, each celebrity was mentioned in an odd number of magazines and each magazine mentioned an odd number of celebrities.

(a) Prove that m and c are either both even or both odd.

(b) In how many ways can that mentioning of celebrities happen? Express this number in terms of c and m.

4. Prove that the polynomial

$$4x^8 - 2x^7 + x^6 - 3x^4 + x^2 - x + 1$$

has no real root.

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big((x-y)^2\big) = x^2 - 2yf(x) + \big(f(y)\big)^2.$$

6. In $\triangle ABC$, $AC > BC$. On the circumcircle of $\triangle ABC$, let D be the midpoint of arc AB containing C. Let E be the point on AC such that $DE \perp AC$.

Prove that $AE = EC + CB$.

7. Prove that there do not exist $w, x, y, z \in \mathbb{N}$ such that

$$x^2 = 10w - 1$$
$$y^2 = 13w - 1$$
$$z^2 = 85w - 1.$$

8. The country of Senso Unico has an airline whose flight routes are arranged as follows:

(i) whenever there is a direct route from city A to city B, there is no direct route from city B to city A;

(ii) there is a route out of every city in Senso Unico;

(iii) whenever there is a direct route from city A to city C, there is a city B such that there is a direct route from city A to city B and a direct route from city B to city C.

The population of Senso Unico is very proud to have the smallest number of cities that allows such an arrangement.

Determine this number.