The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2001 Problems with Solutions to Problem 8

1. Let $L(n)$ be the least common multiple of $1, 2, \ldots, n$.

Determine all prime number pairs (p, q) such that $q = p + 2$ and $L(q) > qL(p)$.

2. Let $\triangle BCA$ be isosceles with $BC = CA$, and circumcentre O. Let P, Q, R be points on AB, BC, CA, respectively, such that $PQ \parallel AC$ and $PR \parallel BC$.

Prove that the quadrilateral $CROQ$ is cyclic.

3. A town has c celebrities and m magazines. One week, each celebrity was mentioned in an odd number of magazines and each magazine mentioned an odd number of celebrities.

(a) Prove that m and c are either both even or both odd.

(b) In how many ways can that mentioning of celebrities happen? Express this number in terms of c and m.

4. Prove that the polynomial

$$4x^8 - 2x^7 + x^6 - 3x^4 + x^2 - x + 1$$

has no real root.

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big((x-y)^2\big) = x^2 - 2yf(x) + \big(f(y)\big)^2.$$

6. In $\triangle ABC$, $AC > BC$. On the circumcircle of $\triangle ABC$, let D be the midpoint of arc AB containing C. Let E be the point on AC such that $DE \perp AC$.

Prove that $AE = EC + CB$.

7. Prove that there do not exist $w, x, y, z \in \mathbb{N}$ such that

$$\begin{aligned} x^2 &= 10w - 1 \\ y^2 &= 13w - 1 \\ z^2 &= 85w - 1. \end{aligned}$$

8. The country of Senso Unico has an airline whose flight routes are arranged as follows:

(i) whenever there is a direct route from city A to city B, there is no direct route from city B to city A;

(ii) there is a route out of every city in Senso Unico;

(iii) whenever there is a direct route from city A to city C, there is a city B such that there is a direct route from city A to city B and a direct route from city B to city C.

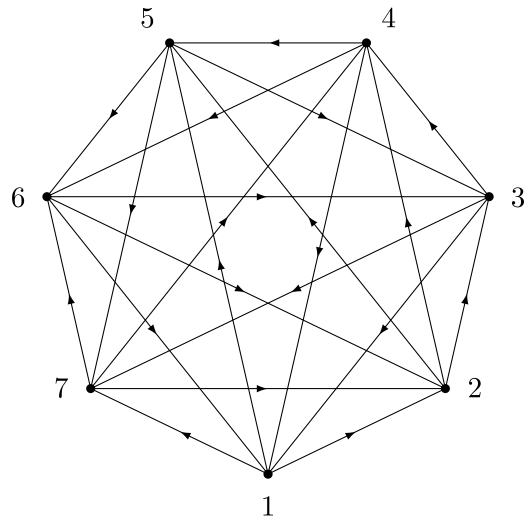
The population of Senso Unico is very proud to have the smallest number of cities that allows such an arrangement.

Determine this number.

Solution. Consider a city A. By (ii), there is a direct route from A to some city C, in which case, by (iii) there is a city B, such that there are direct routes from A to B and from B to C. But, now, since there is a direct route from A to B, (iii) implies there is a city D, say, such that there are direct routes from A to D and from D to B. Also, $D \neq C$, since otherwise we would have direct routes from C to B and from B to C, contradicting (i). Thus any city has at least 3 direct routes out of it.
Thus, assuming there are n cities, then we have $\geq 3n$ flight routes. But (i) implies the number of flight routes is not greater than the number of pairs of cities, i.e.

$$\begin{aligned} 3n &\leq \tfrac{1}{2}n(n-1) = \tbinom{n}{2} \\ \therefore 6 &\leq n-1 \\ \therefore n &\geq 7. \end{aligned}$$

Now, if $n = 7$ is possible, we have $3n = 21 = \binom{7}{2}$, and hence without directions, the graph with cities as vertices and flight routes as edges (noting that (i) implies there is at most one edge between vertices), would be a *complete* graph on 7 vertices, i.e. a K_7 graph. Since counting just the three outward paths for each of the vertices, accounts for all the edges of a K_7 graph, and each vertex is connected by a total of six edges to the other six vertices, the number of inward paths to each vertex must also be three.



We claim that the digraph shown satisfies (i)–(iii). (Note that a *directed edge* is called an *arc*.)
Certainly, between each pair of vertices there is just one arc, and so (i) is satisfied.
Also, each vertex has exactly three outgoing arcs, and hence there is at least the one required by (ii).
Now we determine a criterion for checking (iii).
Suppose the outgoing arcs from A are to X, Y and Z. Then, considering the outward arc $A \to Y$, (iii) says there is a B such that $A \to B \to Y$ is a directed path, which means B must be one of X or Z; without loss of generality, take $X = B$.
Similarly, considering the outward arc $A \to Z$, (iii) implies either $A \to X \to Z$ or $A \to Y \to Z$ is a directed path; without loss of generality, suppose it is $A \to Y \to Z$. Now consider the outward arc $A \to X$. Again (iii) implies either $A \to Y \to X$ or $A \to Z \to X$ is a directed path. But we already have the arc $X \to Y$, and so $Z \to X$ is forced, and hence we have a cycle: $X \to Y \to Z \to X$. Thus, if the outgoing arcs from A are to X, Y and Z, then a cycle such as $X \to Y \to Z \to X$ is forced by (iii). Moreover, if there is such a cycle then (iii) is satisfied.
Checking, we have:

For 1, $1 \to 2$, $1 \to 5$ and $1 \to 7$ are arcs, and $2 \to 5 \to 7 \to 2$ is a cycle.
For 2, $2 \to 3$, $2 \to 4$ and $2 \to 5$ are arcs, and $3 \to 4 \to 5 \to 3$ is a cycle.
For 3, $3 \to 4$, $3 \to 1$ and $3 \to 7$ are arcs, and $4 \to 1 \to 7 \to 4$ is a cycle.
For 4, $4 \to 5$, $4 \to 6$ and $4 \to 1$ are arcs, and $5 \to 6 \to 1 \to 5$ is a cycle.
For 5, $5 \to 3$, $5 \to 7$ and $5 \to 6$ are arcs, and $3 \to 7 \to 6 \to 3$ is a cycle.
For 6, $6 \to 1$, $6 \to 2$ and $6 \to 3$ are arcs, and $1 \to 2 \to 3 \to 1$ is a cycle.
For 7, $7 \to 2$, $7 \to 4$ and $7 \to 6$ are arcs, and $2 \to 4 \to 6 \to 2$ is a cycle.

Thus, (iii) is also satisfied, and hence $n = 7$ is possible.
Hence Senso Unico has 7 cities.