The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2002 Problems

1. Let $m, n \in \mathbb{N}$ such that $2001m^2 + m = 2002n^2 + n$.

Prove that $m - n$ is a perfect square.

2. Determine all triples (u, v, w) of real numbers satisfying

(i) $u + v + w = 38$,

(ii) $uvw = 2002$,

(iii) $0 < u \leq 11$, $w \geq 14$.

3. A line through a vertex of a triangle is called a *dividend* if it cuts the triangle into two triangles with equal perimeters.

Prove that the dividends of a triangle concur.

4. Determine the largest $n \in \mathbb{N}$ for which there exists a set S with exactly n numbers such that

(i) each number in S is a positive integer not exceeding 2002;

(ii) if $a, b \in S$, then $ab \notin S$.

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(2002x - f(0)\big) = 2002x^2 \text{ for all } x \in \mathbb{R}.$$

6. Let $ABCD$ be a rectangle, and let E, F be points on BC, CD, respectively, such that $\triangle AEF$ is equilateral.

Prove that the area of $\triangle ECF$ equals the sum of the areas of $\triangle ABE$ and $\triangle AFD$.

7. Let $n, q \in \mathbb{Z}$ such that $n \geq 5$ and $2 \leq q \leq n$. Prove that $q - 1$ divides

$$\left\lfloor \frac{(n-1)!}{q} \right\rfloor.$$

8. Let $a, b, c, A, B, C, D \in \mathbb{R}$ such that

$$(Ax + B)(Cx + D) = ax^2 + bx + c \text{ for all } x \in \mathbb{R}.$$

Prove that at least one of a, b, c is $\geq \frac{4}{9}(A + B)(C + D)$.