The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2003 Problems

1. Determine all triples (p, q, r) of positive integers that satisfy:

(i) $p(q - r) = q + r$,

(ii) p, q and r are prime numbers.

2. Determine all functions f that are defined for all real numbers $x \neq 0, 1$, with real numbers as their values, and which satisfy

$$f(x) + \frac{1}{2x} f\left(\frac{1}{1-x}\right) = 1.$$

3. Let ABC be a triangle such that $\angle ACB = 2\angle ABC$, and let D be a point in the interior of ABC satisfying $AD = AC$ and $DB = DC$.

Prove that $\angle BAC = 3\angle BAD$.

4. Let

$$p(x) = x^{2003} + a_{2002}x^{2002} + a_{2001}x^{2001} + \cdots + a_2x^2 + a_1x + a_0,$$

where $a_0, a_1, \ldots, a_{2002}$ are integers. Let $q(x) = p(x)^2 - 25$.

Prove that there are not more than 2003 distinct integers m such that $q(m) = 0$.

5. After several kilometres of a televised bicycle race along a straight stretch of road on the Nullarbor, the favourite Andrew pulled well ahead of the rest of the field closely followed by Brenda and then Chris. For the remainder of the race those three were ahead of the rest and, although they frequently changed places, at no time were all three abreast. During the finish a thunderstorm caused the TV signal to drop out, and when it came back on the race was over. The frustrated viewers only heard that the leading position changed 19 times while the third position changed 17 times and that Brenda came third.

Who won the race and why?

6. Let AD be a median of $\triangle ABC$. Let point E lie on AD (extended if necessary) such that $CE \perp AD$. Suppose that $\angle ACE = \angle ABC$.

Prove that either $AB = AC$ or $\angle BAC = 90°$.

7. Let $a_1, a_2, a_3, \ldots$ be a sequence defined by

(i) $a_1 = 0$,

(ii) *either* $a_{i+1} = a_i + 1$ *or* $a_{i+1} = -a_i - 1$, for each $i \geq 0$.

An example is $0, 1, 2, 3, -4, -3, 2, \ldots$.

Prove that $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \geq -\dfrac{1}{2}$, for all positive integers n.

8. Let S be any sequence of n letters ($n \geq 1$) not more than 10 of which are different, e.g. MATHEMATICIANS or GOOLLLDDMMMMEDALLLLLSYESYESYES.

Prove that each letter of the sequence can be replaced by a single decimal digit such that

(i) different letters are replaced by different digits,

(ii) the first letter of the sequence is replaced by a digit other than 0,

(iii) the resulting n-digit number is a multiple of 9.