The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2003 Problems with Some Solutions

1. Determine all triples (p, q, r) of positive integers that satisfy:

(i) $p(q - r) = q + r$,
(ii) p, q and r are prime numbers.

Solution. Let p, q, r be prime, and $p(q - r) = q + r$. Then

$$\begin{aligned} pq - pr &= q + r \\ pq - q &= r + pr \\ q(p - 1) &= r(p + 1) \end{aligned}$$

Suppose $p = 2$. Then $q = 3r$, so that q is not prime $\lightning$.
$\therefore p \neq 2$. So p is odd and so $2 \,|\, p - 1$ and $2 \,|\, p + 1$. Thus, for some $k \in \mathbb{N}$ we have

$$\begin{aligned} p - 1 &= 2k \\ p + 1 &= 2(k + 1) \\ 2kq &= 2(k + 1)r \\ kq &= (k + 1)r \\ k(q - r) &= r \end{aligned}$$

Now, since r is prime, *either* $k = 1, q - r = r$ *or* $k = r, q - r = 1$.
If $k = 1, q - r = r$ then $q = 2r$ so that q is not prime $\lightning$.
$\therefore k = r, q - r = 1$, and hence $q = 3, r = 2$, since 2 and 3 are the only consecutive primes.
Now $p - 1 = 2k = 2r = 4$. Thus $p = 5$.
So there is only one triple (p, q, r) satisfying the given conditions, namely $(p, q, r) = (5, 3, 2)$.

2. Determine all functions f that are defined for all real numbers $x \neq 0, 1$, with real numbers as their values, and which satisfy

$$f(x) + \frac{1}{2x} f\Big(\frac{1}{1 - x}\Big) = 1.$$

Solution. The arguments of f in the given relation are x and $1/(1 - x)$.
Let the function $g(x) = 1/(1 - x)$. Now observe that

$$\begin{aligned} x &\overset{g}{\longmapsto} \frac{1}{1 - x} \\ &\overset{g}{\longmapsto} \frac{1}{1 - 1/(1 - x)} = \frac{1 - x}{1 - x - 1} = \frac{x - 1}{x} = 1 - \frac{1}{x} \\ &\overset{g}{\longmapsto} \frac{1}{1 - (1 - 1/x)} = x. \end{aligned}$$

Taking the domain of g to be $\mathbb{R} \setminus \{0, 1\}$ ensures both $g(x)$ and $g \circ g(x)$ are defined.

The given relation must hold when x is replaced by $g(x)$, and when x is replaced by $g \circ g(x)$. So the given relation implies itself and two further relations are satisfied for all $x \in \mathbb{R} \setminus \{0, 1\}$:

$$f(x) + \frac{1}{2x} f\Big(\frac{1}{1-x}\Big) = 1 \tag{1}$$

$$f\Big(\frac{1}{1-x}\Big) + \frac{1-x}{2} f\Big(\frac{x-1}{x}\Big) = 1 \tag{2}$$

$$f\Big(\frac{x-1}{x}\Big) + \frac{x}{2(x-1)} f(x) = 1 \tag{3}$$

Now let's arrange to eliminate the terms involving $f\big(g(x)\big)$ and $f\big(g \circ g(x)\big)$ from (1)–(3):

$$\begin{aligned} 2x \cdot (1) - (2) + \frac{1-x}{2} \cdot (3): \quad \Big(2x + \frac{1-x}{2} \cdot \frac{x}{2(x-1)}\Big) f(x) &= 2x - 1 + \frac{1-x}{2} \\ \frac{7x}{4} f(x) &= \frac{3x-1}{2} \\ f(x) &= \frac{2(3x-1)}{7x} = \frac{2}{7}\Big(3 - \frac{1}{x}\Big). \end{aligned}$$

Now we check that this is indeed a solution:

$$\begin{aligned} \text{LHS of (1)} &= \frac{2(3x-1)}{7x} + \frac{1}{2x} \cdot \frac{2}{7}\Big(3 - \frac{1}{1/(1-x)}\Big) \\ &= \frac{2(3x-1)}{7x} + \frac{1}{7x}\big(3 - (1-x)\big) \\ &= \frac{6x - 2 + 2 + x}{7x} = 1 = \text{RHS of (1)}, \qquad \text{as required.} \end{aligned}$$

Thus, $f(x) = \frac{2}{7}(3 - 1/x)$ is the only function satisfying the given condition.

3. Let ABC be a triangle such that $\angle ACB = 2\angle ABC$, and let D be a point in the interior of ABC satisfying $AD = AC$ and $DB = DC$.

Prove that $\angle BAC = 3\angle BAD$.

4. Let

$$p(x) = x^{2003} + a_{2002}x^{2002} + a_{2001}x^{2001} + \cdots + a_2x^2 + a_1x + a_0,$$

where $a_0, a_1, \ldots, a_{2002}$ are integers. Let $q(x) = p(x)^2 - 25$.

Prove that there are not more than 2003 distinct integers m such that $q(m) = 0$.

Solution. Assume, for a contradiction, that $q(x)$ has at least 2004 distinct integer zeros. Then, since

$$q(x) = p(x)^2 - 25 = \big(p(x) - 5\big)\big(p(x) + 5\big),$$

by the Pigeon Hole Principle, one of $p(x) - 5$ or $p(x) + 5$ has at least 1002 distinct integer zeros.

Suppose, it is $p(x) - 5$ that has at least 1002 distinct integer zeros, and let 1002 of those, possibly more, zeros be $x_1, x_2, \ldots, x_{1002}$. Then

$$p(x) - 5 = s(x) \prod_{i=1}^{1002} (x - x_i)$$

for some polynomial $s(x)$, by the Factor Theorem. Now, suppose $m \in \mathbb{Z}$ is not a zero of $p(x) - 5$ (such an m exists since $p(x) - 5$ having degree 2003 has only a finite number of (integer) zeros). Then $s(m) \neq 0$, since m is not a zero of $p(x) - 5$. Also, $s(x)$ has integer coefficients, since $p(x)$ and hence $p(x) - 5$ has integer coefficients, and since $p(x) - 5$ when divided by each $x - x_i$ leaves a factor with integer coefficients. Thus $|s(m)| \geq 1$. Also, none of the $m - x_i$ is zero, since m is not a zero of $p(x) - 5$. But, since the x_i are distinct, so are the $m - x_i$. Thus, the smallest the product

$$\left|\prod_{i=1}^{1002} (x - x_i)\right|$$

can be is

$$|-501 \cdot -500 \cdots -2 \cdot -1 \cdot 1 \cdot 2 \cdots 500 \cdot 501| = 1^2 \cdot 2^2 \cdots 501^2 > 10.$$

So $|p(m) - 5| > 10$, and hence by the Triangle Inequality, $|p(m)| > 10 - 5 = 5$, and so, again, by the Triangle Inequality, $|p(m) + 5| > 5 - 5 = 0$, so that $p(m) + 5 \neq 0$, i.e. m is also not a zero of $p(x) + 5$.

Thus, we have shown that, if $p(x) - 5$ has at least 1002 distinct zeros, then any integer that is not a zero of $p(x) - 5$, is not a zero of $p(x) + 5$ either. Also, since $p(x) + 5 = p(x) - 5 + 10$, a zero of $p(x) - 5$ cannot simultaneously be a zero of $p(x) + 5$. Thus, all the zeros of $q(x)$ must be zeros of $p(x) - 5$. But this implies polynomial $p(x) - 5$, which has degree 2003, has 2004 zeros, which is impossible.

Similarly, if instead $p(x) + 5$ has at least 1002 distinct zeros, then all the zeros of $q(x)$ must be zeros of $p(x) + 5$, leading to the same contradiction.

Thus there are not more than 2003 distinct integers m such that $q(m) = 0$.

5. After several kilometres of a televised bicycle race along a straight stretch of road on the Nullarbor, the favourite Andrew pulled well ahead of the rest of the field closely followed by Brenda and then Chris. For the remainder of the race those three were ahead of the rest and, although they frequently changed places, at no time were all three abreast. During the finish a thunderstorm caused the TV signal to drop out, and when it came back on the race was over. The frustrated viewers only heard that the leading position changed 19 times while the third position changed 17 times and that Brenda came third.

Who won the race and why?

6. Let AD be a median of $\triangle ABC$. Let point E lie on AD (extended if necessary) such that $CE \perp AD$. Suppose that $\angle ACE = \angle ABC$.

Prove that either $AB = AC$ or $\angle BAC = 90°$.

7. Let $a_1, a_2, a_3, \ldots$ be a sequence defined by

(i) $a_1 = 0$,

(ii) *either* $a_{i+1} = a_i + 1$ *or* $a_{i+1} = -a_i - 1$, for each $i \geq 0$.

An example is $0, 1, 2, 3, -4, -3, 2, \ldots$.

Prove that $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \geq -\dfrac{1}{2}$, for all positive integers n.

8. Let S be any sequence of n letters ($n \geq 1$) not more than 10 of which are different, e.g. MATHEMATICIANS or GOOLLLDDMMMMEDALLLLLSYESYESYES.

Prove that each letter of the sequence can be replaced by a single decimal digit such that

(i) different letters are replaced by different digits,

(ii) the first letter of the sequence is replaced by a digit other than 0,

(iii) the resulting n-digit number is a multiple of 9.

Solution. Let the resulting n-digit number be N, and let $\mathcal{S}(N)$ represent the sum of the digits of N.
Then (iii) is equivalent to saying $N \equiv 0 \pmod 9$, and hence $\mathcal{S}(N) \equiv 0 \pmod 9$, since $\mathcal{S}(N) \equiv N \pmod 9$.
Also, (ii) is irrelevant, since if we have found a solution of the problem ignoring the condition, we only need to swap the our assignments for 0 and 9 to then also satisfy (ii), since such a swap does not change the value of $\mathcal{S}(N)$ modulo 9.
Let ℓ_k be the k^{th} letter, c_k the number of occurrences of ℓ_k in S, and a_k the digit assigned to ℓ_k. We may as well assume there are 10 different letters, some of which may occur 0 times. Thus we must show there is an assignment such that

$$\mathcal{S}(N) = \sum_{k=0}^{9} c_k a_k \equiv 0 \pmod 9,$$

where for convenience we have numbered the assigned digits from $0, \dots, 9$.
We will consider various assignments of the digits $0, 1, \dots, 9$ to $a_0, a_1, \dots, a_9$. After making an "initial" assignment, let $a = a_{k^*}$ be the a_k that has been assigned 9, and its corresponding frequency $c = c_{k^*}$. We then rotate the values of the other a_k among the values $0, \dots, 8$, in this way: $0 \mapsto 1$, $1 \mapsto 2$, …, $8 \mapsto 0$; and note that modulo 9, this is equivalent to adding 1 to each of the assignments 0 to 8, since $9 \equiv 0 \pmod 9$. Let

$$w = \sum_{k=0}^{9} c_k a_k$$

be the "initial" assignment and let π be the permutation representing the rotation of the assignments $0, \dots, 8$. Then

$$\begin{aligned}
\pi(w) &\equiv \sum_{k \neq k^*} c_k(a_k + 1) + ca \pmod 9\\
&\equiv \sum_{k} c_k a_k + \sum_{k \neq k^*} c_k \pmod 9\\
&\equiv w + n - c \pmod 9
\end{aligned}$$

i.e. each rotation of the assignments $0, \dots, 8$, via π, adds $n - c$ modulo 9, so that

$$\pi^i(w) \equiv w + i(n - c) \pmod 9.$$

Now consider three cases.

Case 1: There is k^* such that $(n - c^*_k, 9) = 1$. In this case, we assign $a = a^*_k = 9$, $c = c^*_k$ and since $(n - c^*_k, 9) = 1$, there exists a multiplicative inverse x of $n - c$ modulo 9, so that with $i = -wx \bmod 9$, we have

$$\begin{aligned}\pi^i(w) &\equiv w + i(n-c) \pmod 9\\ &\equiv 0 \pmod 9\end{aligned}$$

and hence there is an assignment of the a_k satisfying (i)–(iii).

This reduces the problem to cases where all the c_k are such that $n - c_k$ is not coprime to 9, and hence $3 \,|\, n - c_k$ ($\iff n \equiv c_k \pmod 3$) for all k. Hence,

$$\begin{aligned}w = \sum_k c_k a_k &\equiv \sum_k n a_k \pmod 3\\ &\equiv n(0+1+2+\cdots+9) \pmod 3\\ &\equiv 0 \pmod 3.\end{aligned}$$

So now we are ready to consider the remaining cases.

Case 2: $n \equiv c_k \pmod 3$ for all k, but there is k^* such that $n - c^*_k \not\equiv 0 \pmod 9$. Again, we assign $a = a^*_k = 9$, $c = c^*_k$ and have after i rotations

$$\pi^i(w) \equiv w + i(n-c) \pmod 9.$$

But $3 \,|\, w$ and $3 \,|\, (n-c)$, and hence $w = 3w'$ and $n - c = 3d$ for some $w', d \in \mathbb{Z}$. Now, $9 \nmid (n-c) = 3d \implies 3 \nmid d$, and hence there exists a multiplicative inverse x of d modulo 3, so that with $i = -w'x \bmod 3$ we have

$$\begin{aligned}w' + id &\equiv 0 \pmod 3\\ \therefore 0 &\equiv 3w' + i \cdot 3d \pmod 9\\ &\equiv w + i(n-c) \pmod 9\\ &\equiv \pi^i(w) \pmod 9\end{aligned}$$

and hence again there is an assignment of the a_k satisfying (i)–(iii).

Case 3: $n \equiv c_k \pmod 9$ for all k. In this case,

$$\begin{aligned}w = \sum_k c_k a_k &\equiv \sum_k n a_k \pmod 9\\ &\equiv n(0+1+2+\cdots+9) \pmod 9\\ &\equiv 0 \pmod 9.\end{aligned}$$

and hence any valid assignment of the a_k satisfying (i)–(ii), satisfies (iii).

So in each case we have shown that there is an assignment of digits to the letters of S that satisfies (i)–(iii).