The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2004 Problems

1. Determine all pairs (a, b) of real numbers for which the equation $x^3 + 3x^2 + ax + b = 0$ has three different real solutions that can be arranged in arithmetic progression, i.e. the third minus the second is equal to the second minus the first.

2. Suppose $0 \leq x \leq a \leq y \leq b \leq z$ and $a + b + x + y + z = 2004$.

Determine, with proof, the minimum possible value of $x + y + z$.
Determine, with proof, the maximum possible value of $x + y + z$.

3. Determine the number of sequences $a_1, a_2, \ldots, a_{2004}$ which are the numbers $1, 2, \ldots, 2004$ in some order and satisfy

$$|a_1 - 1| = |a_2 - 2| = \cdots = |a_{2004} - 2004| > 0.$$

4. Let ABC be an equilateral triangle, and let D be a point on AB between A and B. Next, let E be a point on AC with $DE \parallel BC$. Further, Let F be the midpoint of CD and G be the circumcentre of $\triangle ADE$.

Determine the angles of $\triangle BFG$.

5. Determine all non-negative integers m, n for which $6^m + 2^n + 2$ is a perfect square.

6. Decide whether or not there is a function f defined for all positive integers and taking positive integers as values such that

$$f\big(f(1)\big) = 5, \quad f\big(f(2)\big) = 6, \quad f\big(f(3)\big) = 4, \quad f\big(f(4)\big) = 3, \quad f\big(f(n)\big) = n + 2 \text{ for } n \geq 5.$$

7. A necklace is made from an even number, $n \geq 4$, of beads, each of which is coloured red, blue or green. There is an equal number of blue beads and green beads on the necklace. It is impossible to cut the necklace into two separate strings such that each string contains a positive even number of beads and such that each string contains the same number of blue and green beads.

Find all the possibilities for the number of red beads on the necklace.

8. Let $ABCD$ be a parallelogram. Suppose there exists a point P in the interior of $ABCD$ such that $\angle ABP = 2\angle ADP$ and $\angle DCP = 2\angle DAP$.

Prove that $AB = BP = CP$.