

AMO TRAINING SESSIONS

**Australian Mathematics Olympiad, 2004 Problems
 with Some Solutions**

1. Determine all pairs (a, b) of real numbers for which the equation $x^3 + 3x^2 + ax + b = 0$ has three different real solutions that can be arranged in arithmetic progression, i.e. the third minus the second is equal to the second minus the first.

2. Suppose $0 \leq x \leq a \leq y \leq b \leq z$ and $a + b + x + y + z = 2004$.

Determine, with proof, the minimum possible value of $x + y + z$.

Determine, with proof, the maximum possible value of $x + y + z$.

3. Determine the number of sequences $a_1, a_2, \dots, a_{2004}$ which are the numbers $1, 2, \dots, 2004$ in some order and satisfy

$$|a_1 - 1| = |a_2 - 2| = \dots = |a_{2004} - 2004| > 0.$$

4. Let ABC be an equilateral triangle, and let D be a point on AB between A and B . Next, let E be a point on AC with $DE \parallel BC$. Further, Let F be the midpoint of CD and G be the circumcentre of $\triangle ADE$.

Determine the angles of $\triangle BFG$.

Solution.

We have $\triangle ABC$ with D, E on AB, AC , respectively, such that $DE \parallel BC$, G the circumcentre of $\triangle ADE$, and F the midpoint of CD .

$$\angle ADE = \angle ABC \quad (\text{corresponding angles, } DE \parallel BC)$$

$\angle A$ is common

$$\therefore \triangle ADE \sim \triangle ABC \quad (\text{by the AA Rule})$$

In particular, $\triangle ADE$ is equilateral.

$\therefore G$ is also the incentre and orthocentre of $\triangle ADE$ (for an equilateral triangle, the incentre (intersection of the medians), orthocentre (intersection of the altitudes) and circumcentre (intersection of the perpendicular bisectors of the sides) all coincide).

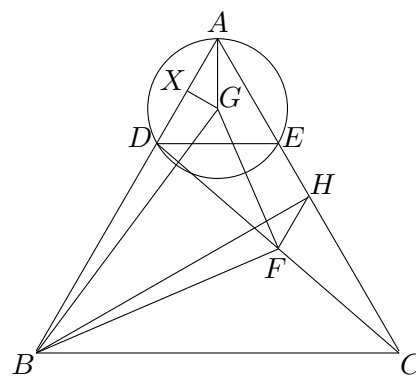
$\therefore AG$ bisects $\angle DAE$. So $\angle BAG = \angle DAG = \frac{1}{2} \cdot 60^\circ = 30^\circ$.

Let H be the midpoint of AC . Then

$$\frac{CF}{CD} = \frac{CH}{CA} = \frac{1}{2}$$

$$\therefore HF \parallel DA$$

$$\therefore \angle FHC = \angle DAC = 60^\circ$$



Now BH is the perpendicular bisector of AC (since for an equilateral triangle the medians coincide with the altitudes).

$$\begin{aligned}\therefore \angle BHC &= 90^\circ \\ \angle BHF &= \angle BHC - \angle FHC \\ &= 90^\circ - 60^\circ = 30^\circ\end{aligned}$$

Let X be the midpoint of AD . Then $\angle AXG = 90^\circ$, and so $\triangle AXG$ has angles $30^\circ, 90^\circ, 60^\circ$ at $\angle A, \angle X, \angle G$, respectively. So $\triangle AXG \sim \triangle BHA$, by the AA Rule.

$$\begin{aligned}\therefore \frac{AB}{HB} &= \frac{GA}{XA} = \frac{AG}{AD/2} = \frac{AG}{FH} \\ \therefore \frac{AB}{AG} &= \frac{HB}{HF} \\ \angle BAG &= \angle BHF = 30^\circ \\ \therefore \triangle BAG &\sim \triangle BHF && \text{(by the SAS Similarity Rule)} \\ \therefore \angle ABG &= \angle HBF \\ \therefore \angle GBF &= \angle ABH = 30^\circ \\ \frac{BF}{BG} &= \frac{BH}{BA} && \text{(since } \triangle BAG \sim \triangle BHF) \\ \therefore \triangle BFG &\sim \triangle BHA && \text{(by the SAS Similarity Rule)}\end{aligned}$$

So $\triangle BHA$ has angles $30^\circ, 90^\circ, 60^\circ$ at $\angle B, \angle H, \angle A$, respectively.

5. Determine all non-negative integers m, n for which $6^m + 2^n + 2$ is a perfect square.

Solution. Let $f(m, n) = 6^m + 2^n + 2 = r^2$, where $m, n, r \in \mathbb{N} \cup \{0\}$. Consider the following cases:

Case 1: $m = 0$. Here $f(m, n) = 6^0 + 2^n + 2 = 2^n + 3$.

Now a perfect square cannot be congruent to 3 modulo 4

(since $x \equiv 0, 1, 2, 3 \pmod{4} \implies x \equiv 0, 1, 0, 1 \pmod{4}$, respectively).

Hence, $n \not\equiv 2$.

For $n = 0$, we have $f(0, 0) = 6^0 + 2^0 + 2 = 4 = 2^2$, is a perfect square.

For $n = 1$, we have $f(0, 1) = 6^0 + 2^1 + 2 = 5$, is not a perfect square.

Case 2: $m \geq 1, n = 0$. Here $f(m, n) = 6^m + 2^0 + 2 = 6^m + 3$.

If $m > 1$ then $3 \mid f(m, n) = r^2 \implies 3 \mid r \implies 3 \mid r^2$, but $3 \nmid 6^m/3 + 1 \#$.

This leaves $m = 1$, for which we have $f(1, 0) = 6^1 + 3 = 9 = 3^2$, a perfect square.

Case 3: $m = 1, n \geq 1$. Here $f(m, n) = 6^1 + 2^n + 2 = 2^n + 8$.

If $n > 3$ then

$$\begin{aligned}2^3 \mid f(m, n) = r^2 &\implies 2^2 \mid r \\ &\implies 2^4 \mid f(m, n), \\ \text{but } 2^4 \mid 2^n \text{ and } 2^4 \nmid 8 &\implies 2^4 \nmid f(m, n) \# \\ \therefore n &\leq 3.\end{aligned}$$

For $n = 1$, $f(1, 1) = 2^1 + 8 = 10$ is not a perfect square.

For $n = 2$, $f(1, 2) = 2^2 + 8 = 12$ is not a perfect square.

However, for $n = 3$, $f(1, 3) = 2^3 + 8 = 16 = 4^2$ is a perfect square.

Case 4: $m \geq 2, n = 1$. Here $f(m, n) = 6^m + 2^1 + 2 = 6^m + 4 = r^2$, so that $2^m 3^m = r^2 - 4 = (r - 2)(r + 2)$.

Now $r - 2 \not\equiv r + 2 \pmod{3}$. So 3^m divides either $r - 2$ or $r + 2$. Also, at least one factor is even, but since they differ by 4, both must be even. Hence one factor is at least $2 \cdot 3^m$ and the other factor is at most 2^{m-1} , i.e.

$$\frac{r+2}{r-2} \geq \frac{2 \cdot 3^m}{2^{m-1}} \geq 4 \cdot \frac{9}{4} = 9$$

and $\frac{r+2}{r-2} = 1 + \frac{4}{r-2} \leq 5$

which is impossible.

Case 5: $m \geq 2, n \geq 2$. Here $2 \mid f(m, n) \implies 2 \mid r^2 \implies 2 \mid r \implies 2^2 \mid f(m, n)$.

With $m \geq 2, n \geq 2$ we have $4 \mid 6^m + 2^n$, so that $4 \nmid 6^m + 2^n + 2 = f(m, n)$, so that $f(m, n)$ cannot be a perfect square.

So we have in all three solutions for (m, n) in the non-negative integers, namely $(m, n) = (0, 0), (1, 0), (1, 3)$.

6. Decide whether or not there is a function f defined for all positive integers and taking positive integers as values such that

$$f(f(1)) = 5, \quad f(f(2)) = 6, \quad f(f(3)) = 4, \quad f(f(4)) = 3, \quad f(f(n)) = n + 2 \text{ for } n \geq 5.$$

Solution. We require $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$f(f(1)) = 5, \tag{1}$$

$$f(f(2)) = 6, \tag{2}$$

$$f(f(3)) = 4, \tag{3}$$

$$f(f(4)) = 3, \tag{4}$$

$$f(f(n)) = n + 2 \text{ for } n \geq 5. \tag{5}$$

Consider the following cases:

Case 1: Suppose $f(3) = 1$. Then

$$f(1) = f(f(3)) = 4, \quad \text{by (3)}$$

$$f(4) = f(f(1)) = 5, \quad \text{by (1)}$$

$$f(5) = f(f(4)) = 3, \quad \text{by (4)}$$

$$1 = f(3) = f(f(5)) = 7 \#, \quad \text{by (5).}$$

$\therefore f(3) \neq 1$.

Case 2: Suppose $f(3) = 2$. Then

$$f(2) = f(f(3)) = 4, \quad \text{by (3)}$$

$$f(4) = f(f(2)) = 6, \quad \text{by (2)}$$

$$f(6) = f(f(4)) = 3, \quad \text{by (4)}$$

$$2 = f(3) = f(f(6)) = 8 \#, \quad \text{by (5).}$$

$\therefore f(3) \neq 2$.

Case 3: Suppose $f(3) = 3$. Then

$$3 = f(3) = f(f(3)) = 4 \#, \quad \text{by (3).}$$

$\therefore f(3) \neq 3$.

Case 4: Suppose $f(3) = 4$. Then

$$\begin{aligned} f(4) &= f(f(3)) = 4, && \text{by (3)} \\ \therefore 4 &= f(4) = f(f(4)) = 3 \#, && \text{by (4)}. \end{aligned}$$

$$\therefore f(3) \neq 4.$$

Case 5: Suppose $f(3) = k$, for some integer $k \geq 5$. Then

$$\begin{aligned} f(k) &= f(f(3)) &= 4, && \text{by (3)} \\ f(4) &= f(f(k)) &= k + 2 \geq 7, && \text{by (5)} \\ f(k + 2) &= f(f(4)) &= 3, && \text{by (4)} \\ k &= f(3) = f(f(k + 2)) &= k + 4 \#, && \text{by (5)}. \end{aligned}$$

$$\therefore f(3) \neq k, \text{ for any } k \geq 5.$$

Thus there is no function $f : \mathbb{N} \rightarrow \mathbb{N}$ that satisfies (1)–(5).

7. A necklace is made from an even number, $n \geq 4$, of beads, each of which is coloured red, blue or green. There is an equal number of blue beads and green beads on the necklace. It is impossible to cut the necklace into two separate strings such that each string contains a positive even number of beads and such that each string contains the same number of blue and green beads.

Find all the possibilities for the number of red beads on the necklace.

8. Let $ABCD$ be a parallelogram. Suppose there exists a point P in the interior of $ABCD$ such that $\angle ABP = 2\angle ADP$ and $\angle DCP = 2\angle DAP$.

Prove that $AB = BP = CP$.

Solution.

Let $ABCD$ be a parallelogram, such that, for a point P inside $ABCD$, $\angle ABP = 2\angle ADP$ and $\angle DCP = 2\angle DAP$.

Construct line ℓ through A parallel to PB .

Construct line m through P parallel to AB .

Let Q be the point of intersection of ℓ and m .

Then $ABPQ$ is a parallelogram.

Also, $QPCD$ is a parallelogram, since $QP = AB = DC$ and $QP \parallel AB \parallel DC$.

Since opposite angles of a parallelogram are equal, $\angle AQP = \angle ABP$.

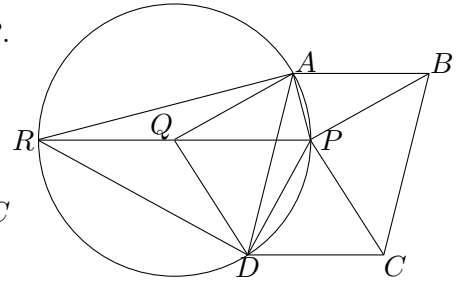
Extend PQ to R such that $QR = QA$. Then $\triangle RQA$ is isosceles.

$$\begin{aligned} \therefore \angle ARQ &= \angle RAQ \\ &= \frac{1}{2}\angle AQP && (\angle AQP \text{ is the sum of its interior opposite angles}) \\ &= \frac{1}{2}\angle ABP \\ &= \angle ADP \end{aligned}$$

\therefore angles at R and D standing on AP are equal.

$\therefore APDR$ is a cyclic quadrilateral.

$\therefore \angle DRP = \angle DAP$ (angles at circumference, standing on DP).



Now,

$$\begin{aligned}\angle QDR &= \angle PDQ - \angle DRQ \\ &= \angle DCQ - \angle DAP && \text{(since } \angle DCP = 2\angle DAP\text{)} \\ &= \angle DAP = \angle DRQ\end{aligned}$$

$\therefore \triangle QRD$ is isosceles, so that Q is the centre of the circumcircle of quadrilateral $APDR$.
Hence $ABPQ$ and $QPCD$ are rhombi, so that $AB = BP = QP = CP$.
 $\therefore AB = BP = CP$, as required.