The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

**AMO TRAINING SESSIONS**

# Australian Mathematics Olympiad, 2005 Problems

**1.** Let $ABC$ be a right-angled triangle with the right angle at $C$. Let $BCDE$ and $ACFG$ be squares external to the triangle. Furthermore, let $AE$ intersect $BC$ at $H$, and let $BG$ intersect $AC$ at $K$.

Find the size of $\angle DKH$.

**2.** Consider a polyhedron whose faces are convex polygons.

Show that it has at least two faces with the same number of edges.

**3.** Let $n$ be a positive integer, and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

Show that

$$\frac{a_1}{a_1^2+1} + \frac{a_2}{a_2^2+1} + \cdots + \frac{a_n}{a_n^2+1} + \leq \frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_n+1}.$$

**4.** Show that, for each positive integer $n$, there exists a positive integer $x$ such that

$$\sqrt{x + 2004^n} + \sqrt{x} = (\sqrt{2005} + 1)^n.$$

**5.** In a multiple choice test there are $q \geq 10$ questions. For each question there are $a > 1$ possible answers, exactly one of which is right. A student who gets $r$ answers right, $w$ answers wrong and does not attempt the other questions will receive a score of

$$\frac{100(r-w)}{q(a-1)}.$$

Determine all different pairs $(q, a)$ such that all possible scores are integers.

**6.** Let $ABC$ be a triangle. Let $D$, $E$, $F$ be points on the line segments $BC$, $CA$ and $AB$, respectively, such that line segments $AD$, $BE$ and $CF$ meet in a single point. Suppose that $ACDF$ and $BCEF$ are cyclic quadrilaterals.

Prove that $AD \perp BC$, $BE \perp AC$ and $CF \perp AB$.

**7.** Let $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ be two sequences of integers such that $a_0 = b_0 = 1$ and, for each non-negative integer $k$,

(i) $a_{k+1} = b_0 + b_1 + b_2 + \cdots + b_k$ and
(ii) $b_{k+1} = (0^2 + 0 + 1)a_0 + (1^2 + 1 + 1)a_1 + \cdots + (k^2 + k + 1)a_k$.

For each positive integer $n$ show that

$$a_n = \frac{b_1 b_2 \ldots b_n}{a_1 a_2 \ldots a_n}.$$

**8.** In an $n \times n$ array, each of $n$ distinct symbols occurs exactly $n$ times. An example with $n = 3$ is shown.
Show that there is a row or column in the array with at least $\sqrt{n}$ distinct symbols.

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 1 | 3 | 2 |
| 2 | 3 | 1 |