

The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2006 Problems

1. Find all solutions to $1 + 5 \times 2^m = n^2$, where m, n are positive integers.
2. Let f be a function taking positive integers to positive integers such that
 - (i) $f(ab) = f(a)f(b)$,
 - (ii) $f(a) < f(b)$ whenever $a < b$,
 - (iii) $f(3) \geq 7$.

Find the smallest value that $f(3)$ can take.

3. Let $PRUS$ be a trapezium with $PR \parallel SU$ such that $\angle SPU = 2\angle UPR$ and $\angle PSR = 2\angle RSU$. Suppose Q is on PR such that QS bisects $\angle PSR$ and, similarly, T is on SU such that TP bisects $\angle SPU$. Let PT meet SQ at E , and let PU meet SR at F . The line through E parallel to SR meets PU at G , and the line through E parallel to PU meets SR at H . Finally, the line through G and H meets PR at K and SU at L .

Prove that $KG = GH = HL$.

4. Let P_1, P_2, \dots, P_n be n different points on a circle. Between each pair of points there is a line segment which is coloured either red or blue. Consider colourings for which $P_i P_j$ is red if and only if $P_{i+1} P_{j+1}$ is blue, for any distinct i and j in the set $\{1, \dots, n\}$. We interpret P_{n+1} as being the same point as P_1 .

a. For which values of n is such a colouring possible?

b. Let a *step* consist of moving along a single red segment from one point to another point.

Show that it is possible to get from each point to any other point in at most three steps.

5. Let $ABCD$ be a square, and let E be a point on its diagonal BD . Suppose that O_1 is the centre of the circle passing through $\triangle ABE$ and O_2 is the centre of the circle passing through $\triangle ADE$.

Show that AO_1EO_2 is a square.

6. For each positive integer n , let $a(n)$ denote the product of all digits of n .

a. Show that $a(n) \leq n$.

b. Find all solutions to the equation $n^2 - 17n + 56 = a(n)$.

7. For each sequence $S = (a_1, a_2, \dots, a_n)$ of non-negative integers let the *offspring* of S be the sequence $T = (b_1, b_2, \dots, b_n)$, where b_i is the number of integers in S to the right of a_i that are less than a_i . For example,

$$\begin{aligned} \text{if } S &= (6, 1, 8, 0, 5, 7, 2, 2, 4, 0, 7, 7, 5), \\ \text{then } T &= (8, 2, 10, 0, 4, 5, 1, 1, 1, 0, 1, 1, 0). \end{aligned}$$

For a given sequence S_0 , let S_1 be the offspring of S_0 , S_2 be the offspring of S_1 , and so on. Show that there exists an integer j such that $S_j = S_{j+1}$.

8. Alice and Bob play the following guessing game. Alice chooses a positive integer a and tells Bob that it is at most 2006. At each turn, Bob chooses a positive integer b and calls it out to Alice. Alice then tells Bob whether or not $a + b$ is prime.

Prove that there is a strategy for Bob to determine a in less than 2006 turns.