The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2006 Problems with Some Solutions and Some Hints

1. Find all solutions to $1 + 5 \times 2^m = n^2$, where m, n are positive integers.

Solution. We start by rearranging:

$$\begin{aligned} 1 + 5 \cdot 2^m &= n^2 \\ 5 \cdot 2^m &= n^2 - 1 \\ &= (n-1)(n+1). \end{aligned}$$

Now, $n-1$ and $n+1$ have the same parity, and since m is positive, we have $2 \mid (n-1)(n+1)$. Hence, in fact both $n-1$ and $n+1$ are even. Thus let $n-1 = 2a$ and $n+1 = 2a+2$. Then

$$\begin{aligned} 5 \cdot 2^m &= 2a(2a+2) \\ 5 \cdot 2^{m-2} &= a(a+1). \end{aligned}$$

Now one of a and $a+1$ is odd and the other even. Thus we have two cases.
Case 1: $a = 5$ and $2^{m-2} = a+1$. Then $2^{m-2} = 5+1 = 6$ which is impossible.
Case 2: $a+1 = 5$ and $2^{m-2} = a$. Then $a = 4 = 2^{m-2}$, whence $m = 4$ and $n = 2a+1 = 9$.
Thus there is exactly one solution, namely $(m, n) = (4, 9)$ for which $n^2 = 81 = 1 + 5 \cdot 2^4$.

2. Let f be a function taking positive integers to positive integers such that

(i) $f(ab) = f(a)f(b)$,
(ii) $f(a) < f(b)$ whenever $a < b$,
(iii) $f(3) \geq 7$.

Find the smallest value that $f(3)$ can take.

Solution. Observe that $f(x) = x^2$ satisfies the criteria. Assume this definition for f. Then

$$\begin{aligned} f(ab) &= (ab)^2 \\ &= a^2 \cdot b^2 = f(a)f(b). \end{aligned}$$

So (i) is satisfied. Condition (ii) says that f should be strictly increasing, which is satisfied by $f(x) = x^2$ for x positive, and in particular for $x \in \mathbb{N}$. Also,

$$f(3) = 3^2 = 9 > 7.$$

So (iii) is satisfied. So, we see that $f(3) = 9$ is possible.
We will be done, if we can show that $f(3) \not< 9$.

We are given $f(a)$ is positive for all $a \in \mathbb{N}$. So, by (i),

$$f(a) = f(a \cdot 1) = f(a)f(1)$$
$$\therefore f(1) = \frac{f(a)}{f(a)} = 1, \text{ since } f(a) \neq 0.$$

So, now (ii) implies $f(2) > f(1)$, i.e. $f(2) \geq 2$.
Suppose $f(2) = 2$. Then, by (i), $f(4) = f(2 \cdot 2) = f(2)^2 = 4$. But then $f(3) < f(4) = 4$ violates (iii), a contradiction.
Suppose $f(2) = 3$. Then by (i), (ii) and (iii),

$$\begin{aligned} f(27) = f(3 \cdot 3 \cdot 3) = f(3)^3 \geq 7^3 = 343 \\ > 243 = 3^5 = f(2)^5 = f(2^5) = f(32), \end{aligned}$$

i.e. we have $f(27) > f(32)$ violating (ii).
Thus $f(2) \geq 4$, so that $f(8) = f(2^3) = f(2)^3 \geq 64$, by (i). So by (i) and (ii),

$$f(3)^2 = f(3^2) = f(9) > f(8) \geq 64 = 8^2$$
$$\therefore f(3) > 8,$$

i.e. $f(3) \not< 9$.
So the least value $f(3)$ can be is 9.

3. Let $PRUS$ be a trapezium with $PR \parallel SU$ such that $\angle SPU = 2\angle UPR$ and $\angle PSR = 2\angle RSU$. Suppose Q is on PR such that QS bisects $\angle PSR$ and, similarly, T is on SU such that TP bisects $\angle SPU$. Let PT meet SQ at E, and let PU meet SR at F. The line through E parallel to SR meets PU at G, and the line through E parallel to PU meets SR at H. Finally, the line through G and H meets PR at K and SU at L.

Prove that $KG = GH = HL$.

4. Let $P_1, P_2, \ldots, P_n$ be n different points on a circle. Between each pair of points there is a line segment which is coloured either red or blue. Consider colourings for which P_iP_j is red if and only if $P_{i+1}P_{j+1}$ is blue, for any distinct i and j in the set $\{1, \ldots, n\}$. We interpret P_{n+1} as being the same point as P_1.

a. For which values of n is such a colouring possible?

b. Let a *step* consist of moving along a single red segment from one point to another point. Show that it is possible to get from each point to any other point in at most three steps.

5. Let $ABCD$ be a square, and let E be a point on its diagonal BD. Suppose that O_1 is the centre of the circle passing through $\triangle ABE$ and O_2 is the centre of the circle passing through $\triangle ADE$.

Show that AO_1EO_2 is a square.

6. For each positive integer n, let $a(n)$ denote the product of all digits of n.

a. Show that $a(n) \leq n$.

b. Find all solutions to the equation $n^2 - 17n + 56 = a(n)$.

7. For each sequence $S = (a_1, a_2, \ldots, a_n)$ of non-negative integers let the *offspring* of S be the sequence $T = (b_1, b_2, \ldots, b_n)$, where b_i is the number of integers in S to the right of a_i that are less than a_i. For example,

if $S = (6, 1, 8, 0, 5, 7, 2, 2, 4, 0, 7, 7, 5)$,
then $T = (8, 2, 10, 0, 4, 5, 1, 1, 1, 0, 1, 1, 0)$.

For a given sequence S_0, let S_1 be the offspring of S_0, S_2 be the offspring of S_1, and so on.

Show that there exists an integer j such that $S_j = S_{j+1}$.

8. Alice and Bob play the following guessing game. Alice chooses a positive integer a and tells Bob that it is at most 2006. At each turn, Bob chooses a positive integer b and calls it out to Alice. Alice then tells Bob whether or not $a + b$ is prime.

Prove that there is a strategy for Bob to determine a in less than 2006 turns.