The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2007 Problems

1. In $\triangle ABC$ let AB be the shortest side. Let the midpoints of BC and AC be X and Y, respectively. Suppose P is on AC such that PX is perpendicular to BC. The circle passing through A, B and P meets the side BC again at Q.

Prove that QY is perpendicular to AC.

2. Let p be an odd prime. Determine all pairs (m, n) of positive integers that satisfy

$$(p-1)(p^n+1) = 4m(m+1).$$

3. Consider an $m \times n$ rectangle divided into mn unit squares using $m-1$ horizontal grid lines and $n-1$ vertical grid lines. The rectangle is to be cut into mn unit squares in stages as follows.

The first stage is to cut the rectangle into two pieces along a grid line. Each subsequent stage consists of choosing one or more pieces and cutting each of these into two along one of its grid lines.

Find, in terms of m and n, the minimum number of stages required.

4. **a.** Prove that for each positive integer n there exists a real number C_n such that

$$r + r^2 + r^3 + \cdots + r^{2n} \le C_n(1 + r^{2n+1})$$

for all positive real numbers r.

b. For each n, find the minimum value of C_n.

5. Consider a 3×3 square array of distinct positive integers. It is called *special* if the eight products of the numbers in each row, each column and each diagonal are equal. For a special array let the common value of the products be P.

a. Show that there is exactly one such P between 5000 and 6500.

b. Give an example of a special array with this value of P.

6. Let a and b be non-zero real numbers such that $a^2 + b^2 = 1$.

Prove that $|a + \frac{a}{b} + b + \frac{b}{a}| \ge 2 - \sqrt{2}$.

7. Let ABC be an acute angled triangle. For each point M on the segment AC, let S_1 be the circle through A, B and M, and let S_2 be the circle through M, B and C. Let D_1 be the disk bounded by S_1, and let D_2 be the disk bounded by S_2.

Show that the area of the intersection of D_1 and D_2 is smallest when $BM \perp AC$.

8. Let S be a finite set of non-negative real numbers including 0. Suppose that whenever a and b belong to S, then at least one of $a + b$ and $|a - b|$ belongs to S.

Prove that

(i) S consists of exactly four distinct real numbers

or

(ii) $S = \{0, r, 2r, \ldots, nr\}$ for some non-negative integer n and some positive real number r.