

The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS
AMO TRAINING SESSIONS

**Australian Mathematics Olympiad, 2007 Problems
with Some Solutions and Some Hints**

1. In $\triangle ABC$ let AB be the shortest side. Let the midpoints of BC and AC be X and Y , respectively. Suppose P is on AC such that PX is perpendicular to BC . The circle passing through A , B and P meets the side BC again at Q .

Prove that QY is perpendicular to AC .

Solution.

By given construction, $ABQP$ is a cyclic quadrilateral.

$$\begin{aligned} \therefore \angle PAB &= 180^\circ - \angle PQB \\ \angle PQX &= 180^\circ - \angle PQB, \quad (\angle XQB \text{ is a straight angle}) \\ &= \angle PAB. \end{aligned}$$

Y and X are midpoints of CA and CB , respectively.

$$\begin{aligned} \therefore \frac{CY}{CA} &= \frac{CX}{CB} = \frac{1}{2} \\ \angle C &\text{ is common} \\ \therefore \triangle CYX &\sim \triangle CAB, && \text{(ASA Similarity Rule)} \\ \therefore \angle CYX &= \angle CAB = \angle PAB = \angle PQX \\ \angle PYX &= 180^\circ - \angle CYX && (\angle CYP \text{ is a straight angle}) \\ &= 180^\circ - \angle PQX \end{aligned}$$

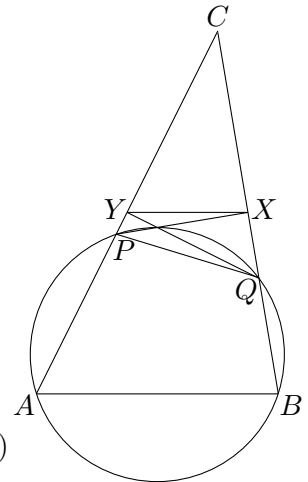
$\therefore PQXY$ is a cyclic quadrilateral

$$\therefore \angle PYQ = \angle PXQ, \quad \text{(angles at circumference of } PQXY \text{ standing on } PQ)$$

$$\angle PXQ = 90^\circ, \quad (PX \perp BC, \text{ given})$$

$$\therefore \angle PYQ = 90^\circ$$

$$\therefore QY \perp AC$$



2. Let p be an odd prime. Determine all pairs (m, n) of positive integers that satisfy

$$(p-1)(p^n+1) = 4m(m+1).$$

Solution. To start, observe that the righthand side can be written as $2m \cdot 2(m+1)$ which is the product of two consecutive even numbers.

Also observe that, if $n = 1$, then the lefthand side becomes $(p-1)(p+1)$, which is also the product of two consecutive even numbers, since p is an odd prime.

So, immediately, we have that $\left(\frac{p-1}{2}, 1\right)$ is a solution.

We might guess at this point that this is in fact all the solutions, and direct our energies to seeking a proof of that.

Another thing we might observe is that

$$(2m + 1)^2 = 4m^2 + 4m + 1 = 4m(m + 1) + 1.$$

This observation will lead us to the complete solution. It follows that:

$$\begin{aligned} (2m + 1)^2 &= 4m(m + 1) + 1 \\ &= (p - 1)(p^n + 1) + 1 \\ &= (p - 1)p^n + p - 1 + 1 \\ &= (p - 1)p^n + p. \end{aligned}$$

Now $p \mid (p - 1)p^n + p$. Hence $p \mid (2m + 1)^2$, and so by Euclid's Lemma, $p \mid (2m + 1)$. But then $p^2 \mid (2m + 1)^2$. So, in fact, $p^2 \mid (p - 1)p^n + p$, whence

$$p \mid (p - 1)p^{n-1} + 1.$$

If $n > 1$, we would have $p \mid (p - 1)p^{n-1}$, from which it would follow that $p \mid (p - 1)p^{n-1} + 1$.

Thus the only solution is the one we observed for $n = 1$, i.e. $\left(\frac{p-1}{2}, 1\right)$ is the only solution.

3. Consider an $m \times n$ rectangle divided into mn unit squares using $m - 1$ horizontal grid lines and $n - 1$ vertical grid lines. The rectangle is to be cut into mn unit squares in stages as follows.

The first stage is to cut the rectangle into two pieces along a grid line. Each subsequent stage consists of choosing one or more pieces and cutting each of these into two along one of its grid lines.

Find, in terms of m and n , the minimum number of stages required.

4. a. Prove that for each positive integer n there exists a real number C_n such that

$$r + r^2 + r^3 + \cdots + r^{2n} \leq C_n(1 + r^{2n+1})$$

for all positive real numbers r .

- b. For each n , find the minimum value of C_n .

Solution. Here are the key steps.

1. Show that $(r^{2n+1-k} - 1)(r^k - 1) \geq 0$ for $k \in \{1, 2, \dots, n\}$ and $r > 0$.
2. Rearrange 1. to get the inequality $r^k + r^{2n+1-k} < 1 + r^{2n+1}$.
3. Use 2. to show $r + r^2 + r^3 + \cdots + r^{2n} \leq n(1 + r^{2n+1})$.
4. For **b.** observe that equality of 3. occurs for $r = 1$.

5. Consider a 3×3 square array of distinct positive integers. It is called *special* if the eight products of the numbers in each row, each column and each diagonal are equal. For a special array let the common value of the products be P .

- a. Show that there is exactly one such P between 5000 and 6500.
- b. Give an example of a special array with this value of P .

Solution. Here's a start. Let the array be

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Multiply the three products aei , def and gac that include e , and divide by the two column products adg and cfi that avoid e . Then we have

$$\frac{(aei)(def)(gac)}{(adg)(cfi)} = P$$

6. Let a and b be non-zero real numbers such that $a^2 + b^2 = 1$.

Prove that $|a + \frac{a}{b} + b + \frac{b}{a}| \geq 2 - \sqrt{2}$.

Solution. Here's a start.

1. The locus of points (a, b) is the unit circle. So $a = \cos(\theta)$ and $b = \sin(\theta)$ for some θ .
2. So $a + b = \cos(\theta) + \sin(\theta)$.
3. Use the identity

$$\sin(\phi + \theta) = \sin(\phi) \cos(\theta) + \cos(\phi) \sin(\theta)$$

with $\sin(\phi) = \cos(\phi) = 1/\sqrt{2}$. Thus

$$\begin{aligned} a + b &= 1 \cdot \cos(\theta) + 1 \cdot \sin(\theta) \\ &= \sqrt{2} \left(\frac{1}{\sqrt{2}} \cos(\theta) + \frac{1}{\sqrt{2}} \sin(\theta) \right) \\ &= \sqrt{2} \sin(45^\circ + \theta) \end{aligned}$$

4. The function \sin is *bounded*: $-1 \leq \sin(\psi) \leq 1$, for any real number ψ .
5. Deduce that $|a + b| \leq \sqrt{2}$.
6. Express $|\frac{a}{b} + \frac{b}{a}|$ in terms of θ and bound it, using essentially the same identity above but with $\phi = \theta$ which gives:

$$\sin(2\theta) = 2 \sin(\theta) \cos(\theta)$$

7. Finish using the Triangle Inequality:

$$|x| - |y| \leq |x + y| \leq |x| + |y|$$

with $x = \frac{a}{b} + \frac{b}{a}$ and $y = a + b$.

7. Let ABC be an acute angled triangle. For each point M on the segment AC , let S_1 be the circle through A , B and M , and let S_2 be the circle through M , B and C . Let D_1 be the disk bounded by S_1 , and let D_2 be the disk bounded by S_2 .

Show that the area of the intersection of D_1 and D_2 is smallest when $BM \perp AC$.

8. Let S be a finite set of non-negative real numbers including 0. Suppose that whenever a and b belong to S , then at least one of $a + b$ and $|a - b|$ belongs to S .

Prove that

(i) S consists of exactly four distinct real numbers

or

(ii) $S = \{0, r, 2r, \dots, nr\}$ for some non-negative integer n and some positive real number r .