The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2008 Problems

1. In K be a circle with PQ as diameter. Let C be a circle with centre on K and with PQ tangent to C.

Prove that the other tangents from P and Q are parallel.

2. Let $f(x) = 5^x$. Determine all real solutions of the equation

$$f\big(x + f(2008)\big) = 2008 - x.$$

3. A positive integer is called *square-free* if it has no factor greater than 1 which is a perfect square.

For each positive integer n, let $f(n)$ be the sum of all square-free factors of n.

Determine all values of n, for which $f(n)/n$ is an integer.

4. Find all positive integers n and all prime numbers p such that the polynomial

$$x^5 + x + p^n$$

can be written as the product of two polynomials with integer coefficients and positive degrees.

5. For each positive integer m, let $F(m)$ be the largest integer such that $10^{F(m)}$ divides $m!$.

Prove that there exists a positive integer n such that for each m

$$\textbf{either} \quad F(m) \leq n \quad \textbf{or} \quad F(m) \geq n + 2008.$$

6. Let $ABCD$ be a convex quadrilateral. Suppose there is a point P on the segment AB with $\angle APD = \angle BPC = 45°$.

If Q is the intersection of the line AB with the perpendicular bisector of CD, prove $\angle CQD = 90°$.

7. Let $A_1A_2A_3$ and $B_1B_2B_3$ be triangles. If

$$p = A_1A_2 + A_2A_3 + A_3A_1 + B_1B_2 + B_2B_3 + B_3B_1, \text{ and}$$
$$q = A_1B_1 + A_1B_2 + A_1B_3 + A_2B_1 + A_2B_2 + A_2B_3 + A_3B_1 + A_3B_2 + A_3B_3,$$

prove that $3p \leq 4q$.

8. A rectangular chessboard has 5 rows and 2008 columns. Each square is painted either red or blue.

Determine the largest integer N which guarantees that, no matter how the chessboard is coloured, there are two rows which have matching colours in at least N columns.