The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

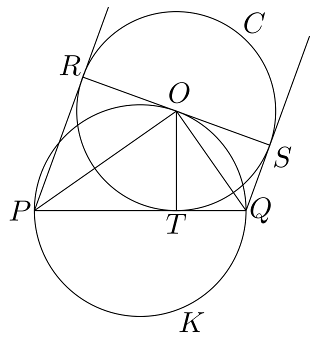
AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2008 Problems with Solutions to Problems 1, 4, 6, 7, and Hints to other Problems

1. In K be a circle with PQ as diameter. Let C be a circle with centre on K and with PQ tangent to C.

Prove that the other tangents from P and Q are parallel.

Solution. Let O be the centre of C, let the other tangent from P to C touch C at R, let the other tangent from Q to C touch C at S, and let PQ touch C at T.



$\angle OTQ = \angle OSQ = 90°$, ($QT$ and ST are tangents to C)

OQ is common

$OT = OS$, (radii of C)

$\therefore \triangle OTQ \cong \triangle OSQ$, by the RHS Rule

$\therefore \angle TOQ = \angle SOQ$

Similarly,

$\triangle OTP \cong \triangle ORP$

$\therefore \angle TOP = \angle ROP$

$\angle POQ = 90°$, since PQ is a diameter of K and O is on K

$$\begin{aligned}\therefore \angle ROS &= \angle ROP + \angle TOP + \angle TOQ + \angle SOQ\\ &= 2(\angle TOP + \angle TOQ)\\ &= 2\angle POQ = 180°\end{aligned}$$

$\therefore RS$ is a straight line with $PR \perp RS \perp QS$

$\therefore PR \parallel QS$

Hence the tangents other than PQ to C from points P and Q are parallel.

2. Let $f(x) = 5^x$. Determine all real solutions of the equation

$$f\big(x + f(2008)\big) = 2008 - x.$$

Solution. *Hint.* $g(x) = f\big(x + f(2008)\big) = 5^{x+c}$ where $c = 5^{2008}$ (constant) is a strictly increasing function of x; $h(x) = 2008 - x$ is a strictly decreasing function of x; there exists x_1 and x_2 such that $h(x_1) > g(x_1)$ and $g(x_2) > h(x_2)$. Putting this together gives the existence of a unique solution, or give a contradiction argument.
Then observe that $x = 2008 - f(2008) = 2008 - 5^{2008}$ satisfies the equation.

3. A positive integer is called *square-free* if it has no factor greater than 1 which is a perfect square.

For each positive integer n, let $f(n)$ be the sum of all square-free factors of n.

Determine all values of n, for which $f(n)/n$ is an integer.

Solution. *Hint.* Try small values of n. Write n in terms of its prime decomposition. Deduce a formula for $f(n)$ and hence a very restricting condition on the prime divisors of $f(n)/n$ that shows that there are only a small finite number of solutions.

4. Find all positive integers n and all prime numbers p such that the polynomial

$$x^5 + x + p^n$$

can be written as the product of two polynomials with integer coefficients and positive degrees.

Solution. Let $q(x) = x^5 + x + p^n = a(x)b(x)$ with $\partial a \le \partial b$ (where ∂u is the *degree* of the polynomial $u(x)$). The either $\partial a = 1$ and $\partial b = 4$ or $\partial a = 2$ and $\partial b = 3$.

Case 1: $\partial a = 2$ and $\partial b = 3$. Then, for some $\alpha, \beta, \gamma, \delta, \varepsilon$,

$$\begin{aligned}
q(x) &= (x^2 + \alpha x + \beta)(x^3 + \gamma x^2 + \delta x + \varepsilon) \\
&= x^5 + \gamma x^4 + \delta x^3 \quad + \varepsilon x^2 \\
&\qquad + \alpha x^4 + \alpha\gamma x^3 + \alpha\delta x^2 + \alpha\varepsilon x \\
&\qquad\qquad + \beta x^3 \quad + \beta\gamma x^2 + \beta\delta x + \beta\varepsilon \\
&= x^5 + (\gamma + \alpha)x^4 + (\delta + \alpha\gamma + \beta)x^3 + (\varepsilon + \alpha\delta + \beta\gamma)x^2 + (\alpha\varepsilon + \beta\delta)x + \beta\varepsilon \\
&= x^5 + \qquad 0 \cdot x^4 + \qquad\qquad 0 \cdot x^3 + \qquad\qquad 0 \cdot x^2 + \qquad 1 \cdot x + p^n
\end{aligned}$$

Equating coefficients we have

$$\begin{aligned}
\gamma + \alpha = 0 &\quad\Longrightarrow\ \gamma = -\alpha \\
\delta + \alpha\gamma + \beta = 0 &\quad\Longrightarrow\ \delta = -\alpha\gamma - \beta \\
&\qquad\quad\ = \alpha^2 - \beta \\
\varepsilon + \alpha\delta + \beta\gamma = 0 &\quad\Longrightarrow\ \varepsilon = -\alpha\delta - \beta\gamma \\
&\qquad\quad\ = \alpha(2\beta - \alpha^2) \\
\alpha\varepsilon + \beta\delta = 1 &\quad\Longrightarrow\ \alpha^2(2\beta - \alpha^2) + \beta(\alpha^2 - \beta) = 1 \\
&\qquad\qquad\qquad \therefore\ 3\alpha^2\beta - \alpha^4 - \beta^2 = 1 \qquad (1) \\
\beta\varepsilon = p^n &\quad\Longrightarrow\ \alpha\beta(2\beta - \alpha^2) = p^n \qquad\qquad (2)
\end{aligned}$$

By (2) each of α and β divide p^n, and so each has magnitude that is a power of p. However, p cannot divide both of α and β, since otherwise p would divide the righthand side of (1). So, one of α and β is ± 1.

Subcase (i): $\alpha = \pm 1$. Then by (1),

$$\begin{aligned}
3\beta - 1 - \beta^2 &= 1 \\
\beta^2 - 3\beta + 2 &= 0 \\
(\beta - 2)(\beta - 1) &= 0
\end{aligned}$$

so that $\beta = 1$ or 2; substitution in (2), give $p^n = \pm 1$ or ± 6, respectively. Either way, p^n is not a power of a prime (contradiction).

Subcase (ii): $\beta = \pm 1$. Then by (1),

$$\pm 3\alpha^2 - \alpha^4 - 1 = 1$$
$$\alpha^4 \mp 3\alpha^2 + 2 = 0$$
$$(\alpha^2 \mp 2)(\alpha^2 \mp 1) = 0$$

so that $\alpha^2 = \pm 1$ or ± 2. Since α is an integer, this gives $\alpha = \pm 1$, only, which was the premise of *Subcase (i)*. So again we have a contradiction.

So we cannot have $\partial a = 2$ and $\partial b = 3$.

Case 2: $\partial a = 1$ and $\partial b = 4$. Let $a(x) = x - \kappa$. Then since $a(x)$ is a linear factor of $q(x)$, by the Factor Theorem,

$$q(\kappa) = \kappa^5 + \kappa + p^n = 0$$
$$\kappa(\kappa^4 + 1) = -p^n. \qquad (3)$$

But $\gcd(\kappa, \kappa^4 + 1) = 1$, whereas each factor of the lefthand side of (3), namely κ and $\kappa^4 + 1$, has absolute value necessarily a power of p. Thus κ or $\kappa^4 + 1$ is ± 1.

Subcase (i): $\kappa^4 + 1 = \pm 1$. Since $\kappa^4 + 1$ cannot be negative we have $\kappa^4 + 1 = 1$ which implies $\kappa = 0$ and consequently $-p^n = 0$ (contradiction).

Subcase (ii): $\kappa = \pm 1$. Since $\kappa^4 + 1$ is positive and $\kappa(\kappa^4 + 1) == -p^n < 0$, we have $\kappa = -1$ which implies $\kappa^4 + 1 = 2$ and consequently $-p^n = 2$, i.e.

$$p = 2, n = 1.$$

Indeed, with $n = 1$ and $p = 2$,

$$x^5 + x + 2 = (x + 1)(x^4 - x^3 + x^2 - x + 2).$$

Thus there is exactly one solution $(n, p) = (1, 2)$ for which

$$x^5 + x + p^n$$

can be written as the product of two positive-degree polynomials over the integers.

5. For each positive integer m, let $F(m)$ be the largest integer such that $10^{F(m)}$ divides $m!$.

Prove that there exists a positive integer n such that for each m

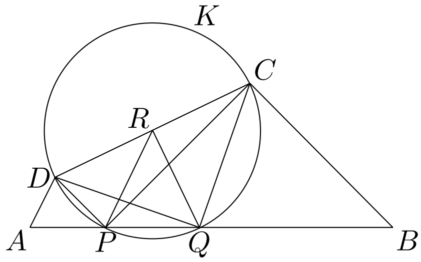
$$\textbf{either} \quad F(m) \leq n \quad \textbf{or} \quad F(m) \geq n + 2008.$$

Solution. *Hint.* In general, $N! = N \cdot (N-1)!$. Use this to deduce an expression for $F(10^{2008})$ and observe that F is monotonic increasing.

6. Let $ABCD$ be a convex quadrilateral. Suppose there is a point P on the segment AB with $\angle APD = \angle BPC = 45°$.

If Q is the intersection of the line AB with the perpendicular bisector of CD, prove $\angle CQD = 90°$.

Solution. Let R be the midpoint of CD. Thus RQ is the perpendicular bisector of CD.



$$\begin{aligned}\angle DPC &= 180^\circ - \angle APD - \angle BPC\\ &= 90^\circ\end{aligned}$$

$\therefore P$ lies on the circle with diameter CD

Call this circle K. Then

D, P, C lie on K which has centre R and radius RC.

Let PB intersect K at Q'. Then

$\angle Q'PC = \angle BPC = 45^\circ$ and

$\angle Q'RC = 2\angle Q'PC = 90^\circ$, angles at circumference and centre standing on common chord $Q'C$

$\therefore Q'$ lies on perpendicular bisector of CD and on AB.

Thus Q' is the intersection of the line AB with the perpendicular bisector of CD.

$\therefore Q' = Q$ lies on K

$\therefore \angle CQD = 90^\circ$, (angle in a semicircle).

Alternative Method. Let $\theta = \angle RPC$. Then

$\angle RCP = \theta$, since $RP = RC$ (radii of K), so that $\triangle PRC$ is isosceles

$\therefore \angle DRP = 2\theta$, (sum of interior opposite angles of $\triangle PRC$)

$$\begin{aligned}\therefore \angle PRQ &= 180^\circ - \angle DRP - \angle CRQ\\ &= 90^\circ - 2\theta\\ \therefore \angle RQP &= 180^\circ - \angle PRQ - \angle RPQ\\ &= 180^\circ - (90^\circ - 2\theta) - (45^\circ + \theta)\\ &= 45^\circ + \theta = \angle RPQ\end{aligned}$$

$\therefore \triangle PRQ$ is isosceles

$\therefore RP = RQ$

$\therefore Q$ lies on K, since $RP = RQ$ is a radius of K

$\therefore \angle CQP = 90^\circ$

since $\angle CQP$ is an angle in a semicircle, as before.

7. Let $A_1A_2A_3$ and $B_1B_2B_3$ be triangles. If

$$p = A_1A_2 + A_2A_3 + A_3A_1 + B_1B_2 + B_2B_3 + B_3B_1, \text{ and}$$
$$q = A_1B_1 + A_1B_2 + A_1B_3 + A_2B_1 + A_2B_2 + A_2B_3 + A_3B_1 + A_3B_2 + A_3B_3,$$

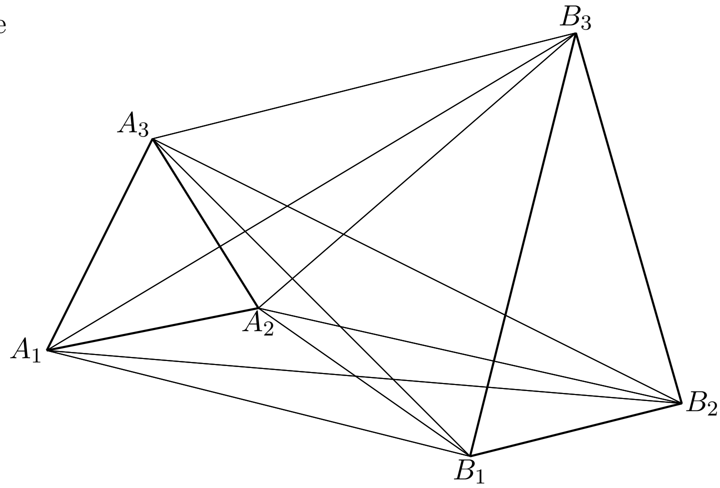
prove that $3p \le 4q$.

Solution. For convenience, let $A_4 = A_1$ and $B_4 = B_1$, so that we may write p and q in Σ notation as follows

$$p = \sum_{i=1}^{3}(A_iA_{i+1} + B_iB_{i+1}) \text{ and } q = \sum_{i=1}^{3}\sum_{j=1}^{3} A_iB_j.$$

Now, by the Triangle Inequality, we have for each i and each j,

$$A_iA_{i+1} \le A_iB_j + B_jA_{i+1} \text{ and}$$
$$B_iB_{i+1} \le B_iA_j + A_jB_{i+1}.$$



Thus we have

$$3A_iA_{i+1} \le \sum_{j=1}^{3}(A_iB_j + B_jA_{i+1}) \text{ and}$$
$$3B_iB_{i+1} \le \sum_{j=1}^{3}(B_iA_j + A_jB_{i+1})$$

$$\begin{aligned}
\therefore\ 3p &= \sum_{i=1}^{3} 3(A_iA_{i+1} + B_iB_{i+1}) \\
&\le \sum_{i=1}^{3}\sum_{j=1}^{3}(A_iB_j + B_jA_{i+1} + B_iA_j + A_jB_{i+1}) \\
&= 4\sum_{i=1}^{3}\sum_{j=1}^{3} A_iB_j, \quad \text{since each of the sums } \textstyle\sum_{i,j} A_iB_j,\ \sum_{i,j} B_jA_{i+1},\ \sum_{i,j} B_iA_j,\ \sum_{i,j} A_jB_{i+1} \text{ covers each of the 9 different } (A_i, B_j) \text{ pairs exactly once and } A_iB_j = B_jA_i \\
&= 4q.
\end{aligned}$$

8. A rectangular chessboard has 5 rows and 2008 columns. Each square is painted either red or blue.

Determine the largest integer N which guarantees that, no matter how the chessboard is coloured, there are two rows which have matching colours in at least N columns.

Solution. *Hint.* Use the Pigeon Hole Principle to show $N \ge 804$. Then show $N \le 804$. Finally, construct an example to show $N = 804$ is indeed the required solution.